Mathematics 136 – Calculus 2
Lab Day 3: Environmental Modeling (Tropical Forests Forever?)
April 11, 2014

Goals

The logistic differential equation

(1) $$\frac{dQ}{dt} = kQ\left(1 - \frac{Q}{M}\right)$$

can be used to model any quantity Q that tends to grow close to exponentially when $Q << M$, but that has a maximum sustainable value M. We have seen that the general solution has the form

(2) $$Q(t) = \frac{M}{1 + ae^{-kt}}$$

where a is a constant that can be determined if we also know an initial condition $Q(0) = Q_0$.

In this lab, we will use the solutions of the logistic equation from (2) above to study an environmental modeling problem in which we analyze several different *harvesting strategies* for tropical hardwood trees.

Note: This lab is adapted from a lab project that accompanies the text *Quantitative Reasoning and the Environment* by Greg Langkamp and Joseph Hull.

Background

Compared to other types of ecosystems, tropical forests contain the greatest amount of plant material per unit area. This is often referred to as the plant *biomass*. Biomass is defined as the total mass of organisms (both living and dead) in a given area or volume. In this project, to simplify, plant biomass will be measured by the carbon content of the plants.

The mean plant biomass on 1 square meter of a mature tropical forest is approximately 20 kg (of carbon). Assume that 80% of the biomass is composed of hardwood trees. Then the average biomass of hardwood trees per hectare (10000 square meters) is approximately

$$(.8) \cdot \frac{20 \cdot 10000}{1000} = 160\text{Mg}$$

(1 Mg = 1 mega-gram = 1000 kg). Assuming that the forest has been undisturbed for a long period of time, this is presumably a close approximation to the carrying capacity of the forest for hardwoods so we will use the value $M = 160$ in our model.

We will use the estimate $k = .1$ for the growth rate constant in the logistic equation. Hence our basic model is

(3) $$\frac{dQ}{dt} = .1Q\left(1 - \frac{Q}{160}\right),$$

where Q represents the hardwood biomass as a function of time.

Maple Notes

Begin by entering

```
with(DEtools);
```

once at the start of the session. You will be using the same `DEplot` command from this package that you used in Lab Day 2. Recall that this can be used both for plotting the slope field of a differential equation and for plotting approximate solutions.

To plot the slope field for an equation

$$y' = f(t, y)$$

you can enter a command of the following format:

```
DEplot(diff(y(t),t)=f(t,y(t)),y(t),t=a..b,y=c..d);
```

To plot solutions of a differential equation, together with the slope field, we can use the same `DEplot` command, but with different options. For example, to plot the solution of the equation $y' = -4ty$ with $y(0) = 8$, for $0 \leq t \leq 1$, you would enter a command like

```
DEplot(diff(y(t),t) = -4*t*y(t),y(t),t=0..1,[[y(0)=8]],linecolor=black);
```

Lab Questions

A) *The clear-cut strategy.* The logging practice in which all, or nearly all, of a forest is removed is called *clear-cutting.* Sometimes all trees are removed, with no action taken to regenerate the forest. Sometimes new seedlings are planted; other times a few trees are left standing to re-seed the forest naturally. Suppose the forest is clear-cut rather thoroughly, leaving only 1.3 Mg of hardwood biomass per hectare. With $Q(0) = 1.3$, let's regrow the forest.

(1) Use Maple to plot the solution of our model differential equation (3) with this initial condition.

(2) Also determine the constant a in the analytic solution of the form in (2) and calculate the values of Q at 10-year intervals for $t = 0$ to $t = 150$ years. You can define a function in Maple to do this for you as follows: with your value of a, define

```
Qsol := t -> 160/(1 + a*exp(-0.1*t));
```

Then `Qsol(30);` will compute the value of the solution at $t = 30$, etc.

(3) The time it takes the forest to grow back to at least 99% of its hardwood carrying capacity is called the *recovery time.* What is the recovery time of this forest?

B) *Selective harvesting.* A second strategy for logging is to harvest only selected trees in the forest. Selective logging usually targets large, mature trees that yield the most lumber for the least effort, or perhaps particularly valuable species of trees if their wood has special properties.

(1) Suppose the 1-hectare plot of the tropical forest forest is mature, with the hardwoods at their carrying capacity. What is the appropriate initial condition together with the logistic equation (3) to model a situation where we begin removing trees.
(2) ("Strategy 1") Suppose 3.2 Mg of hardwood biomass is removed through selective logging each year (about 1 mature tree). Modify the logistic equation (3) to take this harvesting into account.
(3) As in question A (1) above, use Maple to generate a graph showing the solution of your modified equation.
(4) What happens to the hardwood biomass in the long run based on this selective logging strategy.
(5) ("Strategy 2") What if the annual harvest is 6.4 Mg of hardwood biomass per year (about 2 mature trees)? What is the new differential equation? What happens to the biomass in the long run in this case?

C) *Explaining the results seen in question B*

An *equilibrium solution* of an equation like (1) or (3) is a solution that is constant with respect to t (so $Q'(t) = 0$ for all t). Using algebra, determine whether your modified equations from questions B (2) and B (5) have equilibrium solutions and find any such solutions. Explain how this relates to the behavior seen in your graphs in question B.

D) *Long-term strategies for forest management.* From the economic point of view, it would be of interest to compare what the average yields of the forest per year are under clear cutting, and Strategies 1 and 2 from question B.

(1) What is the average yield per year under selective harvesting Strategy 1?
(2) To analyze the clear-cutting strategy, we need to decide how things will work in the long run. Suppose each time the clear-cutting is done, the process works as in question A, then the forest is allowed to rest until it recovers to 99% of its carrying capacity before another clear-cutting is performed. Thus we will have successive cycles of harvest and recovery, repeated over time. How many years does one cycle last (assuming the clear-cutting is done over a short time – much less than one whole year)? How much hardwood is harvested in one cycle? What is the average hardwood harvest per year?
(3) For the selective harvesting Strategy 2, suppose the harvesting is carried out until the remaining biomass is 1.3 Mg per hectare, then the forest is allowed to recover to 99% of its carrying capacity before another cycle of harvesting and recovery starts. How many years does one cycle last in this case? How much hardwood is harvested in one cycle? What is the average hardwood harvest per year?

E) *Sustainability.* Selective harvesting Strategy 1 is *sustainable* because the annual harvest of 3.2 Mg can be maintained each year indefinitely (at least according to the model).

Selective harvesting Strategy 2 is *unsustainable* since if that strategy were followed (without a recovery period), the forest would be completely removed after some number of years.

(1) What is the maximum sustainable annual selective harvesting level? (That is, what is the maximum yearly harvesting level that does not cause the forest to "crash?") You can either do this by experimenting with the harvesting rate and plotting solutions, or via algebra if you are clever!
(2) Many environmentalists think that a sustainable harvesting strategy leaves the forest in a "virgin," or unaffected state. Give an example, based on what you have learned in this project, that indicates this thinking is *false*.
(3) Many people who are "pro-harvest" think that reducing annual harvests today will also lower total harvests in the future and lead to the loss of jobs. Give an example, based on what you have learned in this project, that shows this thinking is also *false*.

Assignment

Writeups due by email to `jlittle@holycross.edu` no later than 5:00pm on Wednesday, April 16.