Mathematics 136 – Calculus 2
Exam 3 – Sample Exam Questions – Solutions
April 22, 2014

I. (A) The arclength is

$$\begin{aligned} L &= \int \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt \\ &= \int_0^2 \sqrt{36t^2 + 36t^4}\, dt \\ &= \int_0^2 6t\sqrt{1+t^2}\, dt, \quad \text{so let } u = 1+t^2; \quad \text{form is } \int u^{1/2}\, du \\ &= 2(1+t^2)^{3/2}\Big|_0^2 \\ &= 2(5\sqrt{5}-1). \end{aligned}$$

(B) The arclength is

$$\begin{aligned} L &= \int \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx \\ &= \int_0^3 \sqrt{1 + \left(\frac{1}{4}(x^2+4)^{1/2}(2x)\right)^2}\, dx \\ &= \int_0^3 \sqrt{\frac{1}{4}x^4 + x^2 + 1}\, dx \quad \text{(a perfect square under the radical)} \\ &= \int_0^3 \frac{1}{2}(x^2+2)\, dx \\ &= \frac{x^3}{6} + x\Big|_0^3 \\ &= \frac{15}{2}. \end{aligned}$$

II. (A) The average value is

$$f_{ave} = \frac{1}{2}\int_0^2 x\sqrt{1+x^4}\, dx$$

To evaluate the integral, use the substitution $u = x^2$ and #21 in the table (or a tangent substitution):

$$\begin{aligned} f_{ave} &= \frac{1}{4}\int_0^4 \sqrt{1+u^2}\, du \\ &= \frac{1}{4}\,\frac{u}{2}\sqrt{1+u^2} + \frac{1}{2}\ln(u+\sqrt{1+u^2})\Big|_0^4 \\ &= \frac{1}{2}\sqrt{17} + \frac{1}{8}\ln(4+\sqrt{17}). \end{aligned}$$

(B) Let the constant density be ρ. The center of mass is at the point $(\overline{x}, \overline{y}) = (M_y/M, M_x/M)$, where

$$M = \int_0^2 \rho x\sqrt{1+x^4}\ dx,$$

and

$$M_x = \int_0^2 \rho \frac{1}{2}(x\sqrt{1+x^4})^2\ dx,$$

while

$$M_y = \int_0^2 \rho x^2\sqrt{1+x^4}\ dx.$$

(C) In order for $f(x)$ to be a probability density function, we need $\int_{-\infty}^{\infty} f(x)\ dx = 1$. Here that means

$$1 = c\int_0^2 x\sqrt{1+x^4}\ dx.$$

From the result in part (A), we see

$$c = \frac{1}{\sqrt{17} + \frac{1}{4}\ln(4+\sqrt{17})}.$$

III. (A) Show that for any constant c, $y = x^2 + \frac{c}{x^2}$ is a solution of the differential equation

$$y' = 4x - \frac{2}{x}y.$$

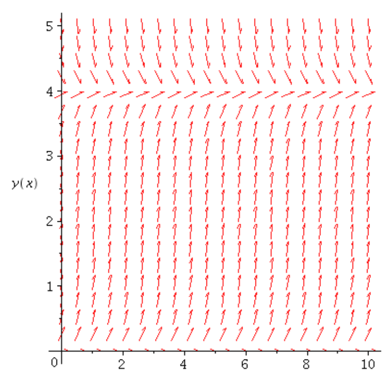
Solution: For $y = x^2 + \dfrac{c}{x^2}$ we have $y' = 2x - \dfrac{2c}{x^3}$ and $4x - \dfrac{2}{x}y = 4x - \dfrac{2}{x}(x^2 + \dfrac{c}{x^2}) = 2x - \dfrac{2c}{x^3}$. Thus $y = x^2 + \dfrac{c}{x^2}$ is a solution to the differential equation $y' = 4x - \dfrac{2}{x}y$.

(B) All parts of this question refer to the differential equation

$$y' = y(4-y)$$

(1) Sketch the slope field of this equation, showing the slopes at points on the lines $y = 0, 1, 2, 3, 4, 5$

Solution:



(2) On your slope field, sketch the graph of the solution of the equation with $y(0) = 1$.

Solution: See figure above.

(3) Use Euler's method to approximate the solution of this equation with $y(0) = 1$ for $0 \le x \le 1$ using $n = 4$.

Solution: We have $\Delta x = 0.25$.

$$\begin{array}{ll} x_0 = 0 & y_0 = 1 \\ x_1 = .25 & y_1 = y_0 + (y_0(4 - y_0))\Delta x = 1 + 3(.25) = 1.75 \\ x_2 = .5 & y_2 = y_1 + (y_1(4 - y_1))\Delta x = 2.734375 \\ x_3 = .75 & y_3 = y_2 + (y_2(4 - y_2))\Delta x = 3.599548340 \\ x_4 = 1 & y_4 = y_3 + (y_3(4 - y_3))\Delta x = 3.959909617 \end{array}$$

(4) This is a separable equation, find the general solution and determine the constant of integration from the initial condition $y(0) = 1$.

Solution: After separating the variables we have $\int \frac{1}{y(4-y)}\, dy = \int dx$. For the integral in y we use partial fractions: $\frac{1}{y(4-y)} = \frac{A}{y} + \frac{B}{4-y}$. We find that $A = B = 1/4$ and thus $\int \frac{1}{y(4-y)}\, dy = \frac{1}{4}\ln|y| - \frac{1}{4}\ln|4-y|$. Therefore, $\frac{1}{4}\ln\left|\frac{y}{4-y}\right| = x + C$. Then $\left|\frac{y}{4-y}\right| = e^{4x}\cdot e^{4C}$ and thus $\frac{y}{4-y} = A\cdot e^{4x}$.

Solving for y, we obtain $y = \frac{4Ae^{4x}}{1+Ae^{4x}}$.

The initial condition $y(0) = 1$ gives $1 = \frac{4A}{1+A}$ and thus $A = 1/3$.

(C) Find the general solutions of the following differential equations

(1) $y' = \frac{y}{x(x+1)}$

Solution: This is a separable differential equation.

We have $\int \frac{dy}{y} = \int \frac{dx}{x(x+1)}$. For the integral on the right we use partial fractions. $\frac{1}{x(x+1)} = \frac{1}{x} - \frac{1}{x+1}$.

Thus $\int \frac{1}{x(x+1)}\, dx = \ln|x| - \ln|x+1| + C = \ln\left|\frac{x}{x+1}\right| + C$.

We have $\ln|y| = \ln\left|\frac{x}{x+1}\right| + C$ and thus $|y| = e^{\ln\left|\frac{x}{x+1}\right| + C} = \left|\frac{x}{x+1}\right|\cdot e^C$.

Therefore $y = A\frac{x}{x+1}$ is the general solution of the given differential equation.

(2) $y' = \dfrac{\sqrt{1-x^2}}{e^{2y}}$.

Solution: This is a separable differential equation.

We have $\int e^{2y}\,dy = \int \sqrt{1-x^2}\,dx$. For the integral on the right we use the trigonometric substitution $x = \sin\theta$, $dx = \cos\theta\,d\theta$. Thus $\int \sqrt{1-x^2}\,dx =$ $\int \sqrt{1-\sin\theta}\cos\theta\,d\theta = \int \cos^2\theta\,d\theta = \int \frac{1+\cos 2\theta}{2}\,d\theta = \frac{1}{2}\theta + \frac{1}{4}\sin 2\theta + C =$

$\frac{1}{2}\theta + \frac{1}{4}2\sin\theta\cos\theta + C = \frac{1}{2}\arcsin x + \frac{1}{2}x\sqrt{1-x^2} + C$

Therefore $\frac{1}{2}e^{2y} = \frac{1}{2}\arcsin x + \frac{1}{2}x\sqrt{1-x^2} + C$ or $e^{2y} = \arcsin x + x\sqrt{1-x^2} + D$ and we have that $y = \frac{1}{2}\ln(\arcsin x + x\sqrt{1-x^2} + D)$ is the general solution to the given differential equation.

(D) Newton's Law of Cooling states that the rate at which the temperature of an object changes is proportional to the difference between the object's temperature and the surrounding temperature. A hot cup of tea with temperature 100°C is placed on a counter in a room maintained at constant temperature 20°C. Ten minutes later the tea has cooled to 76°C. How long will it take to cool off to 45°C? (Express Newton's Law as a differential equation, solve it for the temperature function, then use that to answer the question.)

Solution: Let $T(t)$ denote the temperature of the cup at time t measured in minutes from the time it was placed on the counter. The differential equation modeling this scenario is $\frac{dT}{dt} = k(T-20)$. In fact, this is an initial value problem: $T(0) = 100$ and we have the additional information $T(10) = 76$. This will help us find the constant of proportionality k. The differential equation is separable and we have $\int \frac{dT}{T-20} = \int k\,dt$. Integrating both sides we obtain $\ln|T-20| = kt + C$ and thus $T - 20 = Ae^{kt}$. Therefore $T(t) = 20 + Ae^{kt}$. Since $T(0) = 100$, we have $A = 80$. Since $T(10) = 76$, we have $76 = 20 + 80e^{10k}$. Thus $k = \frac{1}{10}\ln\frac{56}{80}$ and $T(t) = 20 + 80e^{1/10\ln(7/10)t}$. To find the time when the tea has cooled to $45°C$, we sove $20 + 80e^{1/10\ln(7/10)t} = 45$. Thus $e^{1/10\ln(7/10)t} = 25/80 = 5/16$ and the tea will be at $45°C$ after $t = 10\frac{\ln(5/16)}{\ln(7/10)} \approx 32.6$ minutes.

IV. (A) Does the infinite series $\sum_{n=1}^{\infty} n\ln(1+n)$ converge? Why or why not?

Solution: Since $\lim_{n\to\infty} n\ln(1+n) \neq 0$, the series $\sum_{n=1}^{\infty} n\ln(1+n)$ diverges (by the nth Term Divergence Test).

(B) Use the Integral Test to determine whether or not

$$\sum_{k=1}^{\infty} \frac{k}{e^k}$$

converges.

Solution: The function $f(x) = \frac{x}{e^x}$ is continuous and positive. Since $f'(x) = \frac{e^x - xe^x}{e^{2x}} = \frac{e^x(1-x)}{e^{2x}} < 0$ for $x > 1$, f(x) is also decreasing for $x > 1$.
Consider

$$\int_1^{\infty} \frac{x}{e^x}\,dx = \lim_{b\to\infty} \int_1^b xe^{-x}\,dx.$$

Using integration by parts, $u = x$, $du = dx$, $dv = e^{-x}dx$, $v = -e^{-x}$, the improper integral equals $\lim_{b\to\infty}\left(-be^{-b} + e^{-1} + \int_1^b e^{-x}\,dx\right) = \lim_{b\to\infty}\left(-be^{-b} + e^{-1} - e^{-b} + e^{-1}\right)$.
Since

$$\lim_{b\to\infty} e^{-b} = 0$$

and

$$\lim_{b\to\infty} be^{-b} = \lim_{b\to\infty} \frac{b}{e^b} \overset{l'H}{=} \lim_{b\to\infty} \frac{1}{e^b} = 0,$$

the improper integral converges to $2e^{-1}$. By the Integral Test, the series $\sum_{k=1}^{\infty} \frac{k}{e^k}$ converges.

(C) Use the Ratio Test to determine whether or not

$$\sum_{k=0}^{\infty} \frac{3^n}{n!}$$

converges.

Solution:

$$\lim_{n\to\infty} \frac{\frac{3^{n+1}}{(n+1)!}}{\frac{3^n}{n!}} = \lim_{n\to\infty} \frac{3}{n+1} = 0 < 1.$$

By the Ratio Test, the series converges.

(D) Determine (with justification!) whether or not the following series converge:

$$\sum_{k=1}^{\infty} \frac{1}{\sqrt{k}}, \qquad \sum_{n=0}^{\infty} (-1)^n \frac{3^n}{\pi^{2n}}, \qquad \sum_{n=1}^{\infty} \frac{1}{n^{1.01}}.$$

Solution: The series $\sum_{k=1}^{\infty} \frac{1}{\sqrt{k}}$ is the p-series with $p = 1/2$ and thus it diverges.

The series $\sum_{n=0}^{\infty} (-1)^n \frac{3^n}{\pi^{2n}}$ is the geometric series with ratio $\frac{-3}{\pi^2}$. Since the ratio is less than 1 in absolute value, the series converges. (The sum of the series is $\frac{1}{1+\frac{3}{\pi^2}}$.)

The series $\sum_{n=1}^{\infty} \frac{1}{n^{1.01}}$ is the p-series with $p = 1.01$. Since $p > 1$, the p-series converges.

(G) For each of the given power series, find the interval of convergence.

$$f(x) = \sum_{k=0}^{\infty} x^k, \qquad f(x) = \sum_{n=1}^{\infty} \frac{(2x)^n}{\sqrt{n}}, \qquad g(x) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{(x-5)^n}{n \cdot 3^n}.$$

(In particular, give the radius of convergence, and investigate convergence at the endpoints.)

Solution: For $f(x) = f(x) = \sum_{k=0}^{\infty} x^k$, we see that this is a geometric series with ratio x. The series converges if and only if $|x| < 1$. The radius of convergence is 1, the series does not converge for either $x = \pm 1$. The interval of convergence is $(-1, 1)$. (Note: all of this could also be derived through use of the Ratio Test.)

For $g(x) = \sum_{n=1}^{\infty} \frac{(2x)^n}{\sqrt{n}}$, consider the Ratio Test.

$$\lim_{n\to\infty} \left| \frac{\frac{(2x)^{n+1}}{\sqrt{n+1}}}{\frac{(2x)^n}{\sqrt{n}}} \right| = \lim_{n\to\infty} 2|x| \frac{\sqrt{n}}{\sqrt{n+1}} = 2|x|.$$

The series converges if $|x| < 1/2$ and it diverges if $|x| > 1/2$. Since the series is centered at 0 the radius of convergence is 1/2.

If $x = 1/2$, the series equals $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ which is the p-series with $p = 1/2$. Since $p < 1$, the series diverges.

If $x = -1/2$, the series equals $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$. Since the sequence $\frac{1}{\sqrt{n}}$ is decreasing and it converges to 0 as $b \to \infty$, the series converges by the Alternating Series Test.

The interval of convergence for the first series is $[1/2, 1/2)$.

We consider the Ratio Test for $h(x) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{(x-5)^n}{n \cdot 3^n}$.

$$\lim_{n\to\infty} \frac{\frac{|x-5|^{n+1}}{(n+1)3^{n+1}}}{\frac{|x-5|^n}{n\cdot 3^n}} = \lim_{n\to\infty} \frac{|x-5|\cdot n}{3(n+1)} = \frac{|x-5|}{3}.$$

The series converges if $|x-5| < 3$ and it diverges if $|x-5| > 3$. Thus the radius of convergence is 3.

If $x - 5 = 3$, *i.e.*, $x = 8$, the series becomes the alternating harmonic series and it converges.

If $x - 5 = -3$, *i.e.*, $x = 2$, the series equals $g(x) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{(-1)^n}{n} = -\sum_{n=1}^{\infty} \frac{1}{n}$ which is the negative of the harmonic series and thus it diverges.

The interval of convergence for the third series is $(2, 8]$.