Mathematics 136 – Calculus 2
Exam 2 – Review Sheet
March 21, 2014

Sample Exam Questions – Solutions

This list is much longer than the actual exam will be (to give you some idea of the range of different questions that might be asked). Unless otherwise directed, you may use any entry of the Table of Integrals from the text that applies.

I.

(A) Do you need partial fractions to compute

$$\int \frac{t^2 + 1}{t^3 + 3t + 3}\, dt?$$

Explain, and give a simpler method.

No. Just do a u-substitution. $u = t^3 + 3t + 3$, $du = (3t^2 + 3)dt = 3(t^2 + 1)dt$. Then
$$\int \frac{t^2 + 1}{t^3 + 3t + 3}\, dt = \frac{1}{3}\int \frac{1}{u}\, du = \frac{1}{3}\ln|u| + C = \frac{1}{3}\ln|t^3 - 3t + 3| + C.$$

(B) Apply partial fraction decomposition to compute

$$\int \frac{1}{x(x-1)(x+2)}\, dx$$

From the form of the factorization of the denominator we have the following partial fractions:
$$\frac{1}{x(x-1)(x+2)} = \frac{A}{x} + \frac{B}{x-1} + \frac{C}{x+2}.$$
Then $A(x-1)(x+2) + Bx(x+2) + Cx(x-1) = 1$. If $x = 0$, we get $-2A = 1$. Thus, $A = -\frac{1}{2}$. If $x = 1$, we get $3B = 1$. Thus $B = \frac{1}{3}$. If $x = -2$, we get $6C = 1$. Thus $C = \frac{1}{6}$. Therefore
$$\int \frac{1}{x(x-1)(x+2)}\, dx = -\frac{1}{2}\int \frac{1}{x}\, dx + \frac{1}{3}\int \frac{1}{x-1}\, dx + \frac{1}{6}\int \frac{1}{x+2}\, dx =$$
$$-\frac{1}{2}\ln|x| + \frac{1}{3}\ln|x-1| + \frac{1}{6}\ln|x+2| + C.$$

(C) Which trigonometric substitution would you apply to compute $\int \frac{1}{u\sqrt{a^2-u^2}}\, du$? What trigonometric integral do you get after making the substitution? Complete the derivation of the integral.

Use the trigonometric substitution $u = a\sin\theta$, $du = a\cos\theta\, d\theta$. Then

$$\int \frac{1}{u\sqrt{a^2 - u^2}}\, du = \int \frac{1}{a\sin\theta\sqrt{a^2 - a^2\sin^2\theta}}\, a\cos\theta\, d\theta = \int \frac{\cos\theta}{\sin\theta a\sqrt{\cos^2\theta}}\, d\theta =$$

$$\frac{1}{a}\int \frac{\cos\theta}{\sin\theta\cos\theta}\, d\theta = \frac{1}{a}\int \frac{1}{\sin\theta}\, d\theta = \frac{1}{a}\int \csc\theta\, d\theta.$$ Now we use the table of integrals

(formula 15) to obtain that the integral equals $\dfrac{1}{a}\ln|\csc\theta - \cot\theta| + C$. If $u = a\sin\theta$,

then $\csc\theta = a/u$ and $\cot\theta = \dfrac{\sqrt{a^2 - u^2}}{u}$. Therefore

$$\int \frac{1}{u\sqrt{a^2 - u^2}}\, du = \frac{1}{a}\ln\left|\frac{a}{u} - \frac{\sqrt{a^2 - u^2}}{u}\right| + C.$$

II.

(A) Use the midpoint rule, the trapezoidal rule, and Simpson's rule with $n = 4$ to estimate the value of the integral $\int_0^2 \sqrt{1 + x^4}\, dx$, rounding your answers to 6 decimal places.

Solution: With $n = 4$, $\Delta x = \frac{2-0}{4} = .5$, and the intermediate points are $x_0 = 0$, $x_1 = .5$, $x_2 = 1$, $x_3 = 1.5$, and $x_4 = 2$. The midpoints are $m_1 = .25$, $m_2 = .75$, $m_3 = 1.25$, and $m_4 = 1.75$, so the midpoint rule approximation is

$$
\begin{aligned}
\int_0^2 \sqrt{1 + x^4}\, dx \ &\doteq\ \sqrt{1 + (.25)^4}(.5) + \sqrt{1 + (.75)^4}(.5) \\
&\quad + \sqrt{1 + (1.25)^4}(.5) + \sqrt{1 + (1.75)^4}(.5) \\
&\doteq\ (1.001951 + 1.147347 + 1.855103 + 3.221631)(0.5) \\
&\doteq\ 3.613016.
\end{aligned}
$$

The trapezoidal rule approximation is

$$
\begin{aligned}
\int_0^2 \sqrt{1 + x^4}\, dx \ &\doteq\ \frac{(.5)}{2}\left(\sqrt{1 + (0)^4} + 2\sqrt{1 + (.5)^4} + 2\sqrt{1 + (1.0)^4}\right. \\
&\quad \left. + 2\sqrt{1 + (1.5)^4} + \sqrt{1 + (2)^4}\right) \\
&\doteq\ (.25)(1 + 2(1.030776) + 2(1.414213) + 2(2.462214) + 4.123106) \\
&\doteq\ 3.734379.
\end{aligned}
$$

The Simpson's rule approximation is

$$
\begin{aligned}
\int_0^2 \sqrt{1 + x^4}\, dx \ &\doteq\ \frac{(.5)}{3}\left(\sqrt{1 + (0)^4} + 4\sqrt{1 + (.5)^4} + 2\sqrt{1 + (1.0)^4}\right. \\
&\quad \left. + 4\sqrt{1 + (1.75)^4} + \sqrt{1 + (2)^4}\right) \\
&\doteq\ (.166667)(1 + 4(1.030776) + 2(1.414213) + 4(2.462214) + 4.123106) \\
&\doteq\ 3.653470.
\end{aligned}
$$

(B) Which of your answers in part A are overestimates and which are underestimates. Explain how you can tell.

Solution: The function $y = \sqrt{1 + x^4}$ has $y' = \frac{2x^3}{\sqrt{1+x^4}}$ and $y'' = \frac{2x^2(3+x^4)}{(1+x^4)^{3/2}}$. Since both of these are positive on $[0, 2]$, the function is increasing and concave up on this interval. This implies that the midpoint rule is an underestimate and the trapezoidal rule is an overestimate. For Simpson's rule, we do not have enough information to determine whether the approximation is an overestimate or an underestimate.

(C) Which of the estimates in part A would you expect to have the smallest error. Explain. (Review Lab 1 to see the idea here.)

Solution: We expect Simpson's rule to give the most accurate approximation because this Simpson's rule estimate is a weighted average of the midpoint and trapezoidal rule estimates with $n = 2$ (not $n = 4$):

$$S_4 = \frac{1}{3}T_2 + \frac{2}{3}M_2.$$

The weighted average is designed to cancel out the errors. Recall that (at least for large enough n), the trapezoidal rule error has absolute value close to twice the absolute midpoint rule error, but with the opposite sign.

II. For each of the following integrals, say why the integral is improper, determine if the integral converges, and if so, find its value.

A) $\int_1^\infty \frac{1}{\sqrt[5]{x}} \, dx$

Solution: This is an improper integral because of the infinite interval. It converges if

$$\lim_{b\to\infty} \int_1^b x^{-1/5} \, dx$$

is finite. But

$$
\begin{aligned}
\lim_{b\to\infty} \int_1^b x^{-1/5} \, dx &= \lim_{b\to\infty} \frac{5}{4}x^{4/5} \Big|_1^b \\
&= \lim_{b\to\infty} \frac{5}{4}(b^{4/5} - 1) \\
&= +\infty.
\end{aligned}
$$

The limit does not exist as real number, so the integral diverges.

B) $\int_0^2 \frac{dx}{x^2-7x+6}$

Solution: This is an improper integral because the integrand

$$\frac{1}{x^2 - 7x + 6} = \frac{1}{(x-1)(x-6)}$$

3

has an infinite discontinuity at $x = 1$ in the interval $[0, 2]$. It converges if *both limits in*

$$\lim_{b \to 1^-} \int_0^b \frac{dx}{x^2 - 7x + 6} + \lim_{a \to 1^+} \int_a^2 \frac{dx}{x^2 - 7x + 6}$$

exist (are finite). But the first integral gives (by the partial fraction method):

$$
\begin{aligned}
\lim_{b \to 1^-} \int_0^b \frac{dx}{x^2 - 7x + 6} &= \lim_{b \to 1^-} \int_0^b \frac{-1/5}{x - 1} + \frac{1/5}{x - 6} \, dx \\
&= \lim_{b \to 1^-} \left(\frac{-1}{5} \ln |x - 1| + \frac{1}{5} \ln |x - 6| \Big|_0^b \right) \\
&= \lim_{b \to 1^-} \left(\frac{-1}{5} \ln |b - 1| + \frac{1}{5} \ln |b - 6| - \frac{1}{5} \ln(6) \right).
\end{aligned}
$$

The limit does not exist as real number, so the integral diverges.

C) $\int_0^\infty x e^{-3x} \, dx$

Solution: This is an improper integral because of the infinite interval. It converges if

$$\lim_{b \to \infty} \int_0^b x e^{-3x} \, dx$$

is finite. Integrating by parts,

$$
\begin{aligned}
\lim_{b \to \infty} \int_0^b x e^{-3x} \, dx &= \lim_{b \to \infty} \left(\frac{-x}{3} e^{-3x} - \frac{1}{9} e^{-3x} \Big|_0^b \right) \\
&= \lim_{b \to \infty} \frac{-b}{3} e^{-3b} - \frac{1}{9} e^{-3b} + \frac{1}{9} \\
&= \frac{1}{9}
\end{aligned}
$$

(using L'Hopital's Rule to evaluate $\lim_{b \to \infty} \frac{-b}{3e^{3b}} = 0$). The integral converges to $\frac{1}{9}$.

D) For which values of a is $\int_0^\infty e^{ax} \sin(x) \, dx$ convergent? Evaluate the integral for those a.

Solution: This is an improper integral because of the infinite interval. It converges if

$$\lim_{b \to \infty} \int_0^b e^{ax} \sin(x) \, dx$$

is finite. This integral can be evaluated either by integrating by parts twice and solving for the integral, or else consulting # 98 in the table:

$$\int e^{ax} \sin(x) \, dx = \frac{e^{ax}}{a^2 + 1} (a \sin(x) - \cos(x)) + C.$$

Therefore,

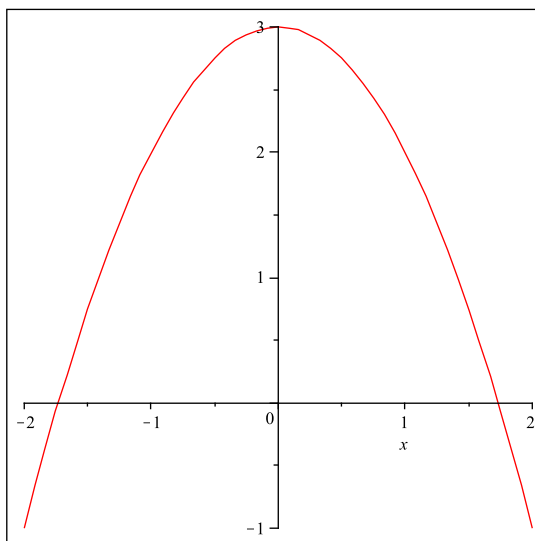$$\int_0^b e^{ax} \sin(x) \ dx = \frac{e^{ab}}{a^2+1}(a\sin(b) - \cos(b)) + \frac{1}{a^2+1}.$$

This has a finite limit as $b \to +\infty$ if and only if $a < 0$. For those a, the first term goes to zero as $b \to +\infty$, and the value of the integral is $\frac{1}{a^2+1}$.

III.

(A) Let R be the region in the plane bounded by $y = 3 - x^2$ and the x-axis.

(1) Sketch the region R.

Solution: The region is bounded above by the parabola $y = 3 - x^2$, and below by the x-axis. It extends from $x = -\sqrt{3}$ to $x = \sqrt{3}$:

(2) Find the area of R.

Solution: The area is

$$A = \int_{-\sqrt{3}}^{\sqrt{3}} 3 - x^2 \ dx = 2 \int_0^{\sqrt{3}} 3 - x^2 \ dx = 2\left(3x - \frac{x^3}{3}\Big|_0^{\sqrt{3}}\right) = 4\sqrt{3}.$$

(3) Find the volume of the solid generated by rotating R about the x-axis.

Solution: Rotating R around the x-axis, the cross-sections are disks with radius
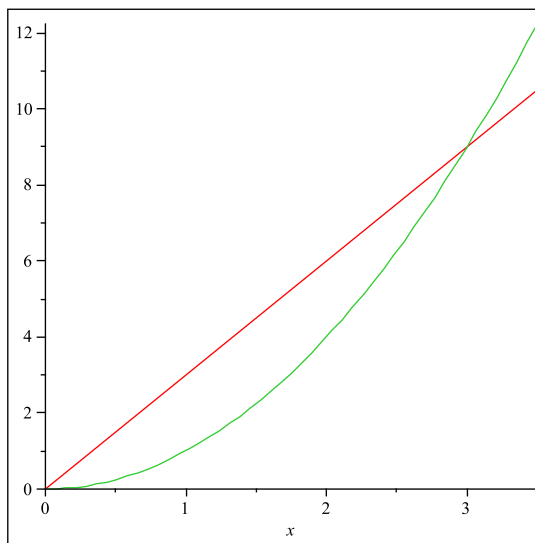
5

$y = 3 - x^2$, so

$$
\begin{aligned}
V &= \int_{-\sqrt{3}}^{\sqrt{3}} \pi(3 - x^2)^2 \ dx \\
&= 2\pi \int_0^{\sqrt{3}} 9 - 6x^2 + x^4 \ dx \\
&= 2\pi \left(9x - 2x^3 + \frac{x^5}{5} \Big|_0^{\sqrt{3}} \right) \\
&= \frac{48\pi\sqrt{3}}{5}.
\end{aligned}
$$

(B) Let R be the region in the plane bounded by $y = 3x$ and $y = x^2$.

  (1) Sketch the region R.

   Solution: The region is bounded above by the line $y = 3x$ and below by the parabola $y = x^2$. It extends from $x = 0$ to $x = 3$:

  (2) Find the area of R.

   Solution: The area is

$$
A = \int_0^3 3x - x^2 \ dx = \left(\frac{3x^2}{2} - \frac{x^3}{3} \Big|_0^3 \right) = \frac{9}{2}.
$$

  (3) Find the volume of the solid generated by rotating R about the x-axis. *Solution:* The cross-sections by planes perpendicular to the x-axis are washers with inner

6

radius $r_{in} = x^2$ and outer radius $r_{out} = 3x$. The volume is

$$
\begin{aligned}
V &= = \int_0^3 \pi(3x)^2 - \pi(x^2)^2 \, dx \\
&= \pi \int_0^3 9x^2 - x^4 \, dx \\
&= \pi \left(3x^3 - \frac{x^5}{5} \Big|_0^3 \right) \\
&= \frac{162\pi}{5}.
\end{aligned}
$$

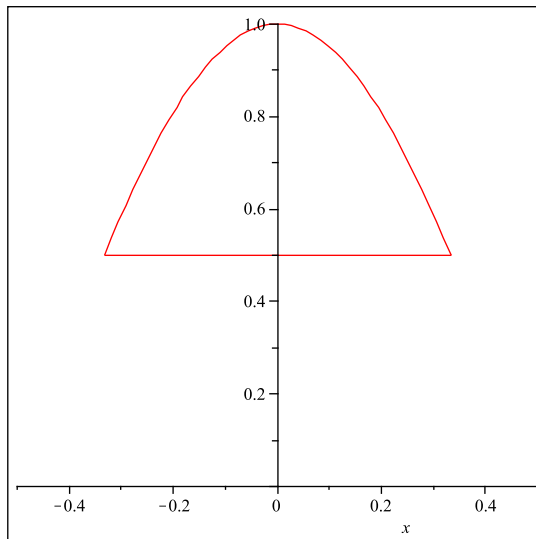(4) Find the volume of the solid generated by rotating R about the y-axis.

Solution: For rotating about the y-axis, we slice horizontally and set up the integral in terms of y. The horizontal cross-sections are washers too, with inner radius $r_{in} = y/3$ (from the line), and outer radius $r_{out} = \sqrt{y}$ (from the parabola). The solid extends from $y = 0$ to $y = 9$ along the y-axis. This gives

$$
\begin{aligned}
V &= = \int_0^9 \pi(\sqrt{y})^2 - \pi(y/3)^2 \, dy \\
&= \pi \int_0^9 y - \frac{y^2}{9} \, dy \\
&= \pi \left(\frac{y^2}{2} - \frac{y^3}{27} \Big|_0^9 \right) \\
&= \frac{27\pi}{2}.
\end{aligned}
$$

(C) Let R be the region in the plane bounded by $y = \cos(\pi x)$, $y = 1/2$, $x = -1/3$ and $x = 1/3$.

(1) Sketch the region R.

Solution: The region is bounded above by an arc of the cosine graph and below by the line $y = 1/2$:

(2) Find the area of R.

Solution: The area is (using symmetry):

$$
\begin{aligned}
A &= \int_{-1/3}^{1/3} \cos(\pi x) - 1/2 \ dx \\
&= 2 \int_{0}^{1/3} \cos(\pi x) - 1/2 \ dx \\
&= 2 \left(\frac{1}{\pi} \sin(\pi x) - \frac{x}{2} \Big|_{0}^{1/3} \right) \\
&= \frac{\sqrt{3}}{\pi} - \frac{1}{3}.
\end{aligned}
$$

(3) Find the volume of the solid generated by rotating R about the x-axis.

Solution: The cross-sections by planes perpendicular to the x-axis are washers with inner radius $r_{in} = 1/2$ and outer radius $r_{out} = \cos(\pi x)$. By symmetry (and integrating with the 1/2-angle formula for $\cos^2(\pi x)$), our volume will be

$$
\begin{aligned}
V &= \int_{-1/3}^{1/3} \pi \cos(\pi x)^2 - \pi \left(\frac{1}{2} \right)^2 \ dx \\
&= 2\pi \int_{0}^{1/3} \frac{1}{2}(1 + \cos(2\pi x)) - \frac{1}{4} \ dx \\
&= 2\pi \left(\frac{x}{4} + \frac{1}{4\pi} \sin(2\pi x) \Big|_{0}^{1/3} \right) \\
&= \frac{\sqrt{3}}{4} + \frac{\pi}{6}.
\end{aligned}
$$

8

IV. The height of a monument is 20m. The horizontal cross-section of the monument at x meters from the top is an isosceles right triangle with legs $x/4$ meters. Find the volume of the monument.

Solution: The area of the cross-section x meters from the top is $A(x) = \frac{1}{2}\left(\frac{x}{4}\right)^2 = \frac{x^2}{32}$. So the volume is

$$V = \int_0^{20} \frac{x^2}{32}\, dx = \left(\frac{x^3}{96}\Big|_0^{20}\right) = \frac{250}{3}.$$

cubic meters.