Mathematics 132 – Calculus for Physical and Life Sciences 2
Exam 1 – Review Sheet
February 14, 2014

Sample Exam Questions- Solutions

I.

(A) Since the interval is $[0, 1]$ and $n = 4$, the sums are

Left endpoints:

$$f(0)(.25) + f(.25)(.25) + f(.5)(.25) + f(.75)(.25) = -.53125$$

Right endpoints:

$$f(.25)(.25) + f(.5)(.25) + f(.75)(.25) + f(1)(.25) = -.78125$$

Midpoints:

$$f(.125)(.25) + f(.375)(.25) + f(.625)(.25) + f(.875)(.25) = -.671875$$

(B) $f'(x) = 2x - 2 \leq 0$ for all $x \in [0, 1]$. Hence f is *decreasing* on this interval. This implies that the left-hand Riemann sum is greater than $\int_0^1 x^2 - 2x\, dx$ (i.e. less negative than the integral), and the right-hand Riemann sum is less (i.e. more negative) than the value of the integral. Using the Evaluation Theorem, we can check this:

$$\int_0^1 x^2 - 2x\, dx = \frac{x^3}{3} - x^2\Big|_0^1 = -\frac{2}{3} \doteq -.66667$$

The left-hand sum is greater (less negative) and the right-hand sum is less (more negative) than this value.

(C) The sum is a right-hand Riemann sum for the function $f(x) = \frac{\cos(x)}{x}$ on the interval $[0, \pi]$. So (if the limit exists), it would equal

$$\int_0^\pi \frac{\cos(x)}{x}\, dx.$$

(In point of fact, we can see with techniques developed later that this is an improper integral *that does not converge* – the limit is not going to be a finite value! Oops!)

II. Let

$$f(x) = \begin{cases} 1 & \text{if } 0 \leq x \leq 3 \\ x - 2 & \text{if } 3 \leq x \leq 5 \\ 13 - 2x & \text{if } 5 \leq x \leq 8 \end{cases}$$

(A) Sketch the graph $y = f(x)$. (Omitted – the graph is made up of segments of three different straight lines.)

In the rest of the parts, $F(x) = \int_0^x f(t)\ dt$, where f is the function from part A.

(B) Assuming $F(0) = 0$, Compute $F(1), F(2), F(3), F(4), F(5), F(6), F(7), F(8)$ given the information in the graph of f.

Using the area interpretation of the definite integral we have

$$F(1) = \int_0^1 f(x)\,dx = 1$$

$$F(2) = \int_0^2 f(x)\,dx = 2$$

$$F(3) = \int_0^3 f(x)\,dx = 3$$

$$F(4) = \int_0^3 f(x)\,dx + \int_3^4 f(x)\,dx = 3 + \frac{3}{2} = \frac{9}{2}$$

$$F(5) = \int_0^4 f(x)\,dx + \int_4^5 f(x)\,dx = \frac{9}{2} + \frac{5}{2} = 7$$

$$F(6) = \int_0^5 f(x)\,dx + \int_5^6 f(x)\,dx = 7 + 2 = 9$$

$$F(7) = \int_0^6 f(x)\,dx + \int_6^{13/2} f(x)\,dx + \int_{13/2}^7 f(x)\ dx = 9 + \frac{1}{4} - \frac{1}{4} = 9$$

$F(8) = \int_0^5 f(x)\,dx + \int_5^{13/2} f(x)\,dx + \int_{13/2}^8 f(x)\,dx = \int_0^5 f(x)\,dx = 7$ (the last two integrals cancel since they represent equal areas with opposite signs).

(C) Are there any critical points of F? If so, find them and say whether they are local maxima, local minima, or neither. If not, say why not.

By the Fundamental Theorem of Calculus, $F'(x) = f(x)$. Since $f(13/2) = 0$, the point $x = 13/2$ is a critical point. Since $F' = f$ changes sign from positive to negative at the critical point, $x = 13/2$ is a local maximum.

(D) Sketch the graph $y = F(x)$ if $F(0) = 0$, and also if $F(0) = 2$. Also omitted.

III. Find the derivatives of the following functions (A) $f(x) = \int_0^x \sin(t)/t\,dt$.

$$f'(x) = \frac{\sin x}{x}$$

(B) $g(x) = \int_5^{x^3} \tan^4(t)\ dt$.

$g(x) = m(x^3)$, where $m(x) = \int_5^x \tan^4(t)\ dt$. Then, $g'(x) = m'(x^3) \cdot 3x^2 = \tan^4(x^3) \cdot 3x^2$.

(C)) $h(x) = \int_{-3x}^{5x} e^{t^2} \sin(t)\ dt$.

$h(x) = n(x) + l(x)$, where $n(x) = \int_{-3x}^{0} e^{t^2}\sin(t)\ dt$ and $l(x) = \int_{0}^{5x} e^{t^2}\sin(t)\ dt$. Then $h'(x) = n'(x) + l'(x) = -(e^{(-3x)^2}\sin(-3x))\cdot(-3) + 5\cdot e^{(5x)^2}\sin(5x) = 3e^{9x^2}\sin(-3x) + 5e^{25x^2}\sin(5x)$.

IV.

(A) Compute $\int 5x^4 - 3\sqrt{x} + e^x + \frac{2}{x}\ dx$

$$\int 5x^4 - 3\sqrt{x} + e^x + \frac{2}{x}\ dx = x^5 - 2x^{3/2} + e^x + 2\ln|x| + C$$

(B) Apply a u-substitution to compute $\int x(4x^2-3)^{3/5}\ dx$

$u = 4x^2 - 3,\ du = 8x\,dx$. Then $\int x(4x^2-3)^{3/5}\ dx = \int \frac{1}{8}u^{3/5}\,dx = \frac{1}{8}\frac{u^{8/5}}{8/5} + C = \frac{5}{64}(4x^2-3)^{8/5} + C$

(C) Apply a u-substitution to compute $\int_1^2 e^{\sin(\pi x)}\cos(\pi x)\ dx$

$u = \sin(\pi x),\ du = \pi\cos(\pi x)$. Then $\int_1^2 e^{\sin(\pi x)}\cos(\pi x)\ dx = \frac{1}{\pi}\int_0^0 e^u\,du = 0$

(D) Do you need partial fractions to compute

$$\int \frac{t^2+1}{t^3+3t+3}\ dt?$$

Explain, and give a simpler method.

No. Just do a u-substitution. $u = t^3 + 3t + 3,\ du = (3t^2+3)dt = 3(t^2+1)dt$. Then $\int \frac{t^2+1}{t^3+3t+3}\ dt = \frac{1}{3}\int \frac{1}{u}\ du = \frac{1}{3}\ln|u| + C = \frac{1}{3}\ln|t^3 - 3t + 3| + C$.

(E) Apply integration by parts to compute $\int x^2e^{-2x}\ dx$

$u = x^2 \qquad du = 2x\,dx$
$dv = e^{-2x} \qquad v = -\frac{1}{2}e^{-2x}$
Then $\int x^2e^{-2x}\ dx = -\frac{1}{2}x^2e^{-2x} + \int xe^{-2x}\ dx$. Integration by parts again.
$u = x \qquad du = dx$
$dv = e^{-2x} \qquad v = -\frac{1}{2}e^{-2x}$
Then $\int x^2e^{-2x}\ dx = -\frac{1}{2}x^2e^{-2x} - \frac{1}{2}xe^{-2x} + \frac{1}{2}\int e^{-2x}\ dx = -\frac{1}{2}x^2e^{-2x} - \frac{1}{2}xe^{-2x} - \frac{1}{4}e^{-2x} + C$

V. Compute each of the integrals below using some combination of basic rules, substitution, integration by parts, the table of integrals, partial fractions, and trigonometric substition. You must show all work for full credit.

(A)

$$\int \frac{e^{\sqrt{\sin(x)}}\cos(x)}{\sqrt{\sin(x)}}\ dx$$

Use the substitution $u = \sqrt{\sin x}, du = \dfrac{1}{2\sqrt{\sin x}} \cos x\, dx$. Then

$$\int \frac{e^{\sqrt{\sin(x)}}\cos(x)}{\sqrt{\sin(x)}}\ dx = 2\int e^u\, du = 2e^u + C = 2e^{\sqrt{\sin x}} + C.$$

(B) Use parts twice (with $u =$ the trig function and $dv = e^x\ dx$, then solve for the desired integral algebraically:

$$\begin{aligned}
\int e^x \sin(2x)\ dx &= e^x \sin(2x) - 2\int e^x \cos(2x)\ dx \\
\int e^x \sin(2x)\ dx &= e^x \sin(2x) - 2\left(e^x \cos(2x) + 2\int e^x \sin(2x)\ dx\right) \\
\int e^x \sin(2x)\ dx &= e^x \sin(2x) - 2e^x \cos(2x) - 4\int e^x \sin(2x)\ dx \\
\therefore 5\int e^x \sin(2x)\ dx &= e^x \sin(2x) - 2e^x \cos(2x), \quad \text{and} \\
\int e^x \sin(2x)\ dx &= \frac{e^x \sin(2x) - 2e^x \cos(2x)}{5} + C.
\end{aligned}$$

(C) Integrate by parts ($u = \tan^{-1}(x), dv = dx$), the resulting integral is $\frac{1}{2}\int \frac{dw}{w}$ for $w = x^2+1$, and use the Evaluation Theorem:

$$\begin{aligned}
\int_0^1 \tan^{-1}\ dx &= x\tan^{-1}(x)\Big|_0^1 - \int_0^1 \frac{x}{x^2+1}\ dx \\
&= \frac{\pi}{4} - \frac{1}{2}\ln(x^2+1)\Big|_0^1 \\
&= \frac{\pi}{4} - \frac{\ln(2)}{2}.
\end{aligned}$$

(D) This reduction formula falls out immediately from the integration by parts formula, letting $u = x^n$ and $dv = e^{ax}\ dx$.

(E) We use the reduction formula from (D) with $n = 4$, $a = -2$, then again with $n = 3, 2, 1$:

$$\begin{aligned}
\int x^4 e^{-2x}\ dx &= \frac{-x^4 e^{-2x}}{2} + 2\int x^3 e^{-2x}\ dx \\
&= \frac{-x^4 e^{-2x}}{2} + 2\left(\frac{-x^3 e^{-2x}}{-2} + \frac{3}{2}\int x^2 e^{-2x}\ dx\right) \\
&\vdots \\
&= -\left(\frac{x^4}{2} + x^3 + \frac{3}{2}x^2 + \frac{3}{2}x + \frac{3}{4}\right)e^{-2x}
\end{aligned}$$