MATH 135 – Problem Set 9 B Solutions

§4.5 / 60. Let $f(x) = x^n e^{-x}$. Then by the product and chain rules,

$$f'(x) = nx^{n-1}e^{-x} - x^n e^{-x} = x^{n-1}e^{-x}(n-x)$$

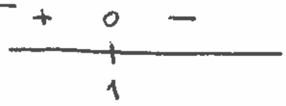
$$f''(x) = n(n-1)x^{n-2}e^{-x} - 2nx^{n-1}e^{-x} + x^n e^{-x}$$

$$= \begin{cases} x^{n-2}e^{-x}\left(x^2 - 2nx + n(n-1)\right) & \text{if } n \geq 2 \\ -e^{-x}(2-x) & \text{if } n = 1 \end{cases}$$

Setting $f'(x) = 0$ we find critical numbers $x = 0$ (if $n > 1$) and $x = n$. The sign of $f'(x)$ for $x < 0$ depends on whether n is even or odd, like this:

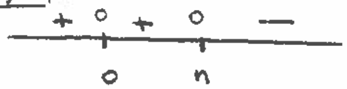
Sign of f':

n=1



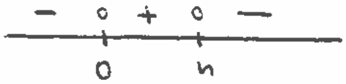
$f'(x) = e^{-x}(1-x)$

n odd, >1:



$f'(x) = x^{n-1}e^{-x}(n-x)$ (note: $n-1$ is even)

n even:



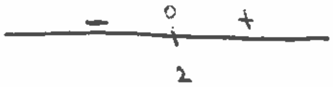
$f'(x) = x^{n-1}e^{-x}(n-x)$ (note: $n-1$ is odd)

If n is even, $\begin{cases} x=0: \\ x=n: \end{cases}$ gives a local minimum of f / maximum

If n is odd $\begin{cases} x=0: & f \text{ has neither a local max nor a local min} \\ x=n: & \text{has a local max} \end{cases}$

Sign of f'': $f''(x) = 0$ if $x = 0$ $(n>2)$ or if $x = n \pm \sqrt{n}$ (quadratic formula on $x^2 - 2nx + n(n-1) = 0$)

n=1

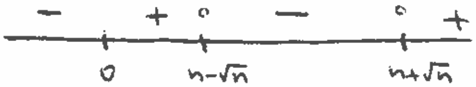


$f''(x) = (x-2)e^{-x}$

n ≥ 2 even

f'' so $n-\sqrt{n}$, $n+\sqrt{n}$ are inflection points, 0 is not

n > 2 odd



f'' 0, $n-\sqrt{n}$, $n+\sqrt{n}$ all inflection points.

Finally, by L'Hôpital's Rule

$$\lim_{x\to+\infty} x^n e^{-x} = \lim_{x\to+\infty} \frac{x^n}{e^x} \qquad \frac{\infty}{\infty}$$

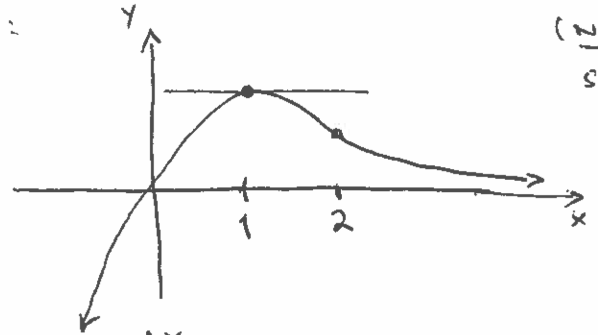
$$= \lim_{x\to+\infty} \frac{n x^{n-1}}{e^x} \qquad \text{still } \frac{\infty}{\infty} \text{ if } n>1$$

$$\vdots$$

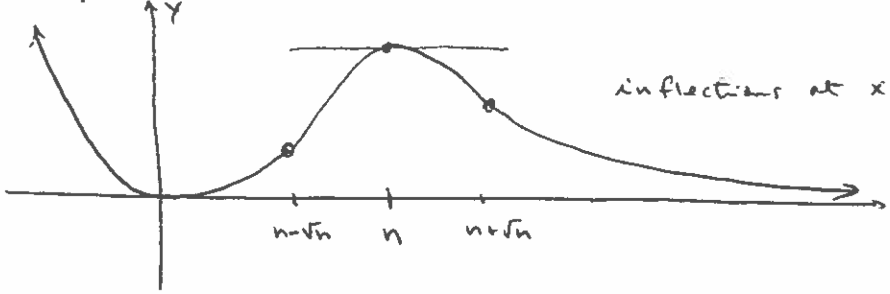
$$= \lim_{x\to\infty} \frac{n(n-1)\cdots(2)(1)}{e^x} = 0.$$

Graphs: (NB: not to scale; these just show general shapes)

n=1 :

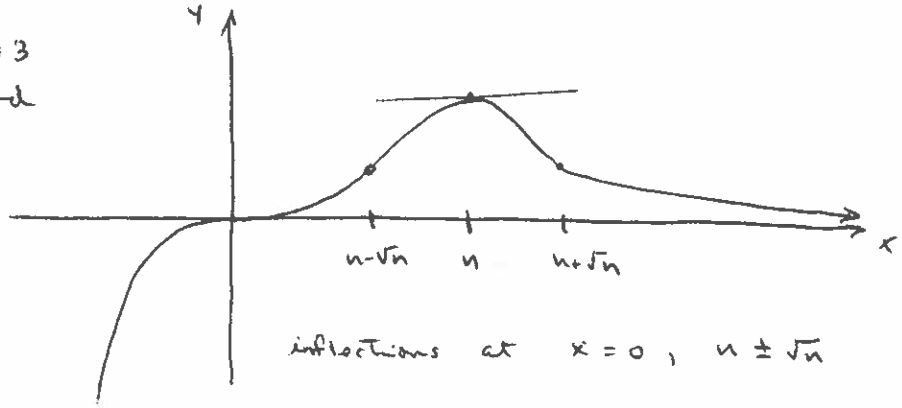


n ≥ 2
even



inflections at x = n ± √n

n ≥ 3
odd



inflections at x = 0, n ± √n

66. (a) $v = \frac{mg}{c}(1 - e^{-ct/m})$ where $c, m > 0$

so $\lim_{t\to\infty} \frac{mg}{c}(1 - e^{-ct/m}) = \frac{mg}{c} - \lim_{t\to\infty} \frac{mg}{c} e^{-ct/m} = \boxed{\frac{mg}{c}}$

("terminal velocity")

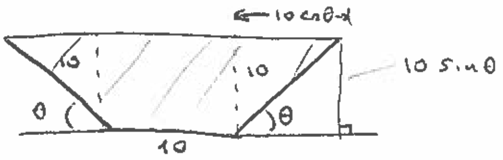
b) $\lim_{c\to 0^+} \frac{mg}{c}(1 - e^{-ct/m})$ is a $\frac{0}{0}$ form.

By L'Hopital, it equals

$$\lim_{c\to 0^+} \frac{-mg\cdot e^{-ct/m}\cdot\left(-\frac{t}{m}\right)}{1} = \lim_{c\to 0^+} gt\, e^{-ct/m} \to \boxed{gt}.$$

Hence v is <u>not</u> <u>bounded</u> for free fall in a vacuum. (If the body can fall for t seconds, the velocity will reach gt) seconds

§4.6/56



<u>Solution 1</u>: the gutter with the maximum capacity is the one with the largest <u>cross-section</u> <u>area</u>, That is the area of this trapezoid

$$A(\theta) = \frac{1}{2}(10 + 10 + 20\cos\theta)(10\sin\theta)$$

$$= 100\sin\theta + 100\sin\theta\cos\theta$$

To maximize $A(\theta)$, we want θ making $A'(\theta) = 0$:

$$A'(\theta) = 100\cos\theta - 100\sin^2\theta + 100\cos^2\theta \quad \text{(product rule)}$$

$$= 100\left[\cos\theta - (1-\cos^2\theta) + \cos^2\theta\right]$$

$$= 100\left[\cos\theta - 1 + 2\cos^2\theta\right]$$

$$= 100\cdot(2\cos\theta - 1)(\cos\theta + 1) = 0$$

when $\cos\theta = \frac{1}{2}$ or $\cos\theta = -1$

$\theta = \frac{\pi}{3}$ or $\theta = \pi$

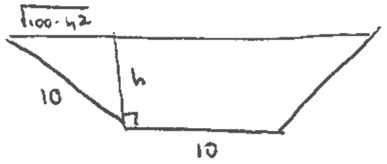
$A(0) = 0$

$A(\frac{\pi}{3}) > 0$ ← max achieved here.

$A(\pi) = 0$

$$\boxed{\theta = \frac{\pi}{3}}$$

Solution 2: We can also set this up in terms of the height of the trapezoid, then solve for θ.



$$A(h) = 10h + \cancel{2}\cdot\frac{1}{\cancel{2}}\sqrt{100-h^2}\cdot h$$

$$= 10h + h\sqrt{100-h^2}$$

then $$0 = A'(h) = 10 + \sqrt{100-h^2} - \frac{h^2}{\sqrt{100-h^2}}$$

$$\Rightarrow \frac{h^2}{\sqrt{100-h^2}} = 10 + \sqrt{100-h^2}$$

so $$h^2 = 10\sqrt{100-h^2} + 100 - h^2$$

$$2h^2 - 100 = 10\sqrt{100-h^2}$$

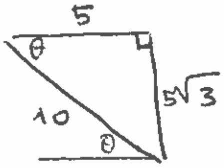
Square: $$4h^4 - 400h^2 + \cancel{10000} = \cancel{10000} - 100h^2$$

so $$4h^4 = 300h^2$$

$$h^2 = 75$$

so $\boxed{h = 5\sqrt{3}}$ (and clearly gives a max.)

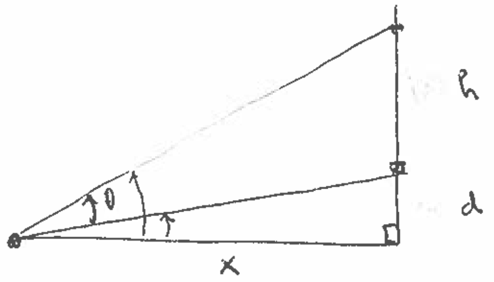
For this h:



$\Rightarrow \boxed{\theta = \frac{\pi}{3}}$ as before

is a "30-60-90" right triangle.

58.



$$\theta = \tan^{-1}\left(\frac{h+d}{x}\right) - \tan^{-1}\left(\frac{d}{x}\right)$$

so $$\frac{d\theta}{dx} = \frac{1}{1+\left(\frac{h+d}{x}\right)^2}\cdot\left(\frac{-(h+d)}{x^2}\right) - \frac{1}{1+\left(\frac{d}{x}\right)^2}\left(-\right.$$

$$= \frac{-(h+d)}{x^2+(h+d)^2} + \frac{d}{x^2+d^2}$$

For a max, we want $\frac{d\theta}{dx} = 0$, so

$$\frac{h+d}{x^2+(h+d)^2} = \frac{d}{x^2+d^2}$$

$$(h+d)x^2 + (h+d)d^2 = dx^2 + d(h+d)^2$$

$$\begin{aligned} \text{so} \quad hx^2 &= d(h+d)^2 - (h+d)d^2 \\ &= dh^2 + 2d^2h + \cancel{d^3} - hd^2 - \cancel{d^3} \\ &= dh^2 + d^2h \end{aligned}$$

$$\therefore \quad x^2 = dh + d^2 = d(h+d)$$

so $x = \sqrt{d(h+d)}$ (can assume $x \geq 0$)

Note $\theta(0) = 0$ (really $\lim_{x\to 0^+} \theta(x) = 0$)

and $\lim_{x\to\infty} \theta(x) = 0$. So the unique critical point we found must be where $\theta(x)$ attains a <u>max</u>.