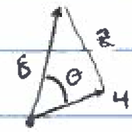
MATH 135 PS 8-B Solutions

4.1/44. Consider the triangle formed by the 2 hands at any time:



By the Law of Cosines,

$z^2 = 64 + 16 - 64\cos\theta = 80 - 64\cos\theta$. So differentiating with respect to t, $2z\frac{dz}{dt} = 64\sin\theta\frac{d\theta}{dt}$. At 1 o'clock, $\theta = \frac{2\pi}{12} = \frac{\pi}{6}$ so $\cos\theta = \frac{\sqrt{3}}{2}$, and $d = \sqrt{80-32\sqrt{3}}$.

We also know $\frac{d\theta}{dt} = -2\pi + \frac{2\pi}{12} = \frac{-11\pi}{6}$ radians/hour. So at that time,

$$\frac{dz}{dt} = \frac{64\cdot\frac{1}{2}\cdot\frac{-11\pi}{6}}{2\sqrt{80-32\sqrt{3}}} \doteq \boxed{-18.6 \text{ mm/hr}}$$

Alternate method: the position of the minute hand's tip t hours after 12 $\frac{\text{am}}{\text{pm}}$ is $(8\cos(2\pi t), 8\sin(2\pi t))$, while the hour hand's tip is at $\left(4\cos\left(\frac{2\pi t}{12}\right), 4\sin\left(\frac{2\pi t}{12}\right)\right) = \left(4\cos\left(\frac{\pi t}{6}\right), 4\sin\left(\frac{\pi t}{6}\right)\right)$. The distance between the tips is

$$z = \sqrt{\left[8\cos(2\pi t) - 4\cos\left(\frac{\pi t}{6}\right)\right]^2 + \left[8\sin(2\pi t) - 4\sin\left(\frac{\pi t}{6}\right)\right]^2}$$

$$= \sqrt{64\left(\cos^2(2\pi t) + \sin^2(2\pi t)\right) + 16\left(\sin^2\left(\frac{\pi t}{6}\right) + \cos^2\left(\frac{\pi t}{6}\right)\right) - 64\cos\left(\frac{11\pi t}{6}\right)}$$

$$= \sqrt{80 - 64\cos\left(\frac{11\pi t}{6}\right)}$$ (by cosine-addition formula)

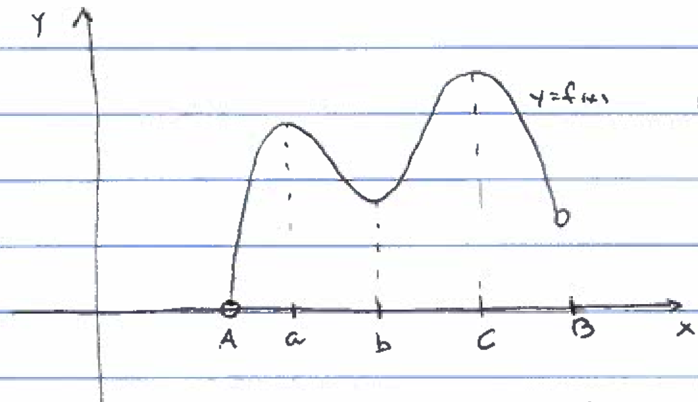
So $$\frac{dz}{dt} = \frac{64\sin\left(\frac{11\pi t}{6}\right)\cdot\frac{11\pi}{6}}{2\sqrt{80-64\cos\left(\frac{11\pi t}{6}\right)}}$$

at 1 pm $\Leftrightarrow$ $t = 1$ hour

$$\frac{dz}{dt} = \frac{64\left(-\frac{1}{2}\right)\cdot\frac{11\pi}{6}}{2\sqrt{80-32\sqrt{3}}} \doteq \boxed{-18.6 \text{ mm/hr}}$$ (as before)

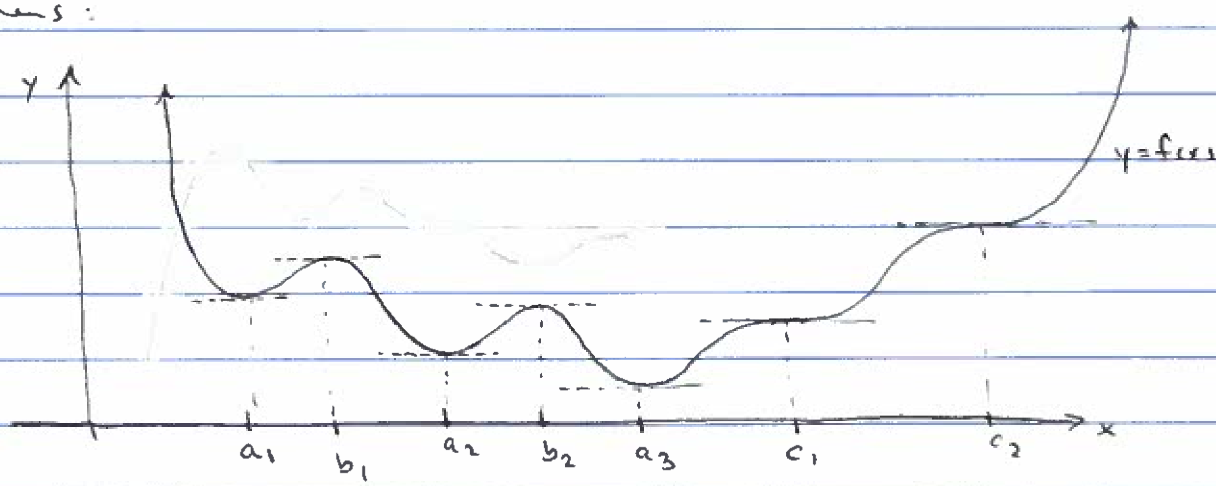
4.2/14

(a) two local maxima, one local minimum, no absolute minimum. One way is to make a function on an open interval



No absolute minimum since $\lim_{x \to A^+} f(x) = 0$ but $f(x) \neq 0$ for all $x \in (A, B)$.

(b) 3 local minima, 2 local maxima, 7 critical numbers:



f has local minima at $x = a_1, a_2, a_3$

maxima at $x = b_1, b_2$

critical numbers at $x = c_1, c_2$ (neither)

Other graphs with some points where $f'(x)$ dne would also be OK. However, vertical asymptotes, end points don't count as critical points.

§4.3 / 72 For $f(x) = cx + \frac{1}{x^2+3}$, we compute

$$f'(x) = c - \frac{2x}{(x^2+3)^2}$$

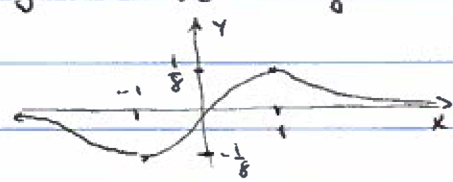
So $f'(x) \geq 0$ for all x and f is increasing as long as c is greater than the max value of $g(x) = \frac{2x}{(x^2+3)^2}$. So we need to determine that max value. Using calculus again,

$$g'(x) = \frac{(x^2+3)^2 \cdot 2 - 2x \cdot 2(x^2+3)\cdot(2x)}{(x^2+3)^4}$$

$$= \frac{6-6x^2}{(x^2+3)^3}$$

— (−1) + (1) — sign g'

By the First Derivative Test, g has a local maximum at $x=1$. $g(1)$ is also the absolute maximum of $g(x)$ since $g(x) < 0$ if $x < 0$ but $g(1) = \frac{2}{16} = \frac{1}{8} > 0$ the graph $y = g(x)$ looks like this



since $\lim_{x \to \pm\infty} g(x) = 0$.

Hence any $c \geq \frac{1}{8}$ will make $f'(x) \geq 0$ for all x (and $f'(x) = 0$ only at $x = 1$ for $c = \frac{1}{8}$). So those $f(x)$ will be increasing for $x \in (-\infty, \infty)$.