MATH 135 - Problem Set 7B Solutions

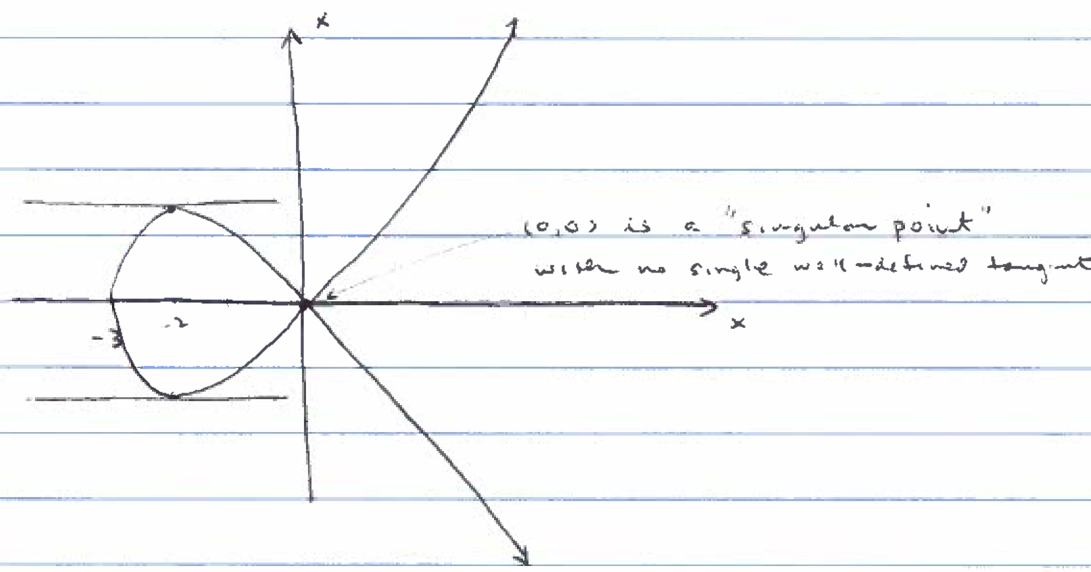
§3.5 / 30

(a) $\boxed{y^2 = x^3 + 3x^2}$, so by implicit differentiation,

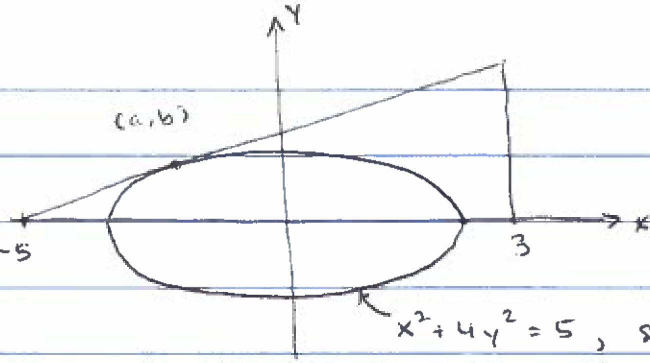
$$2y\frac{dy}{dx} = 3x^2 + 6x \quad \text{and} \quad \frac{dy}{dx} = \frac{3x^2+6x}{2y}.$$

At $(x,y) = (1,-2)$, $\frac{dy}{dx} = -\frac{9}{4}$, so the tangent line is
$y+2 = -\frac{9}{4}(x-1)$ or $\boxed{y = -\frac{9}{4}x + \frac{1}{4}}$.

(b) $\frac{dy}{dx} = 0$ if $0 = 3x^2 + 6x = 3x(x+2)$, $\overbrace{\ }^{\text{so } x=0 \text{ or } x=-2,}$ but $y \neq 0$.
If $x = 0$, then $y = 0$ from the equation of the cubic, so that solution must be discarded. If $x = -2$, then $y^2 = (-2)^3 + 3(-2)^2 = 4$, so $y = \pm 2$. There are two points with horizontal tangents $\boxed{(-2,2),\ (-2,-2)}$
the reason $x = 0$ does not "work" here comes from the geometry of the curve $y^2 = x^3 + 3x^2$:



56.



$x^2+4y^2=5$, so $2x+8y\frac{dy}{dx}=0$

and $\boxed{\frac{dy}{dx} = \frac{-x}{4y}}$

At the point (a,b) on the ellipse where the tangent hits $(-5,0)$, we have $\frac{dy}{dx} = \frac{-a}{4b}$, and (a,b) is also on the tangent line, so $b = \frac{-a}{4b}(a+5)$. this rearranges to $4b^2 = -a^2 - 5a$, so $5 = a^2+4b^2 = -5a$, and $\boxed{a=-1}$. From $a^2+4b^2=5$, then $\boxed{b=+1}$ (the other root $b=-1$ gives a point on the lower half of the ellipse, so it's not relevant.) But then $\frac{dy}{dx} = \frac{1}{4}$ for the tangent line, so the equation is $y = \frac{1}{4}(x+5)$. when $x=3$, $\boxed{y = \frac{1}{4}(3+5) = 2}$.

3.6/28. Find $\frac{dy}{dx}$ if $y = \tan^{-1}\left(\sqrt{\frac{1-x}{1+x}}\right)$

<u>Solution 1:</u> ("Brute force") using $\frac{d}{dx}\tan^{-1}u = \frac{1}{1+u^2}\frac{du}{dx}$

$$\frac{dy}{dx} = \frac{1}{1+\left(\sqrt{\frac{1-x}{1+x}}\right)^2} \cdot \frac{1}{2}\left(\frac{1-x}{1+x}\right)^{-1/2} \cdot \frac{-2}{(1+x)^2}$$, then simplify:

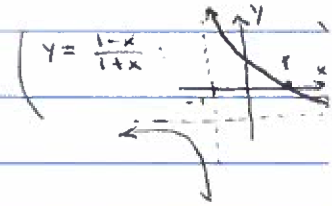
$$= \frac{x+1}{2} \cdot \left(\frac{(1+x)^{1/2}}{(1-x)^{1/2}}\right) \cdot \frac{-1}{(1+x)^2}$$

$$= -\frac{1}{2} \cdot \frac{1}{\sqrt{(1-x)}\cdot\sqrt{1+x}}$$

$$= \boxed{-\frac{1}{2\sqrt{1-x^2}}}$$

Solution 2: ("very clever") Note that in $\sqrt{\frac{1-x}{1+x}}$,

the "inside" is only > 0 if $-1 \le x \le 1$



this says we can rewrite $\boxed{x = \cos 2\theta}$

for $0 \le \theta \le \pi/2$, and then $1 - x = 1 - \cos 2\theta$

$$= (\cos^2\theta + \sin^2\theta) - (\cos^2\theta - \sin^2\theta)$$

$$\boxed{1 - x = 2\sin^2\theta}, \text{ and}$$

$$1 + x = 1 + \cos 2\theta$$

$$\boxed{1 + x = 2\cos^2\theta}$$

So $\sqrt{\frac{1-x}{1+x}} = \sqrt{\frac{\sin^2\theta}{\cos^2\theta}} = \tan\theta$. Hence $\tan^{-1}\left(\sqrt{\frac{1-x}{1+x}}\right) =$

$y = \tan^{-1}(\tan\theta) = \theta = \frac{1}{2}\cos^{-1}(x)$. Thus,

$$\frac{dy}{dx} = \frac{-1}{2\sqrt{1-(x)^2}} = \boxed{\frac{-1}{2\sqrt{1-x^2}}}$$ by formula 2 in §3.6

§3.7/44 By logarithmic differentiation, if

$x^y = y^x$, $y \ln x = x \ln y$ and then

$$\ln(x)\frac{dy}{dx} + \frac{y}{x} = \frac{x}{y}\frac{dy}{dx} + \ln(y)$$

So $$\frac{dy}{dx} = \frac{\left(\ln(y) - \frac{y}{x}\right)}{\left(\ln(x) - \frac{x}{y}\right)} \cdot \frac{xy}{xy}$$

$$\boxed{\frac{dy}{dx} = \frac{xy\ln(y) - y^2}{xy\ln(x) - x^2}}$$

§3.8/28 We are given $f = \frac{1}{2L}\sqrt{\frac{T}{\rho}}$

(a) (i) If T, ρ are constant, $\boxed{\frac{df}{dL} = -\frac{1}{2}\sqrt{\frac{T}{\rho}} \cdot \frac{1}{L^2}}$

ii. If L, ρ constant, $\dfrac{df}{dT} = \dfrac{d}{dT}\left(\dfrac{1}{2L\sqrt{\rho}}\, T^{1/2}\right)$

$$= \frac{1}{2L\sqrt{\rho}} \cdot \frac{1}{2\sqrt{T}}$$

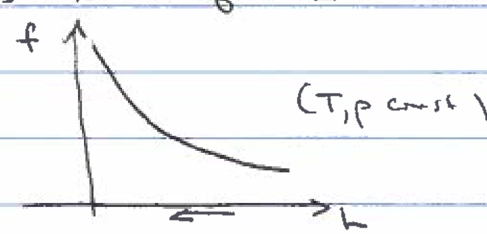
$$= \boxed{\frac{1}{4L\sqrt{\rho}\sqrt{T}}}$$

iii. If L, T constant $\dfrac{df}{d\rho} = \dfrac{d}{d\rho}\left(\dfrac{\sqrt{T}}{2L}\, \rho^{-1/2}\right)$

$$= \frac{\sqrt{T}}{2L} \cdot -\frac{1}{2}\rho^{-3/2}$$

$$= \boxed{\frac{-\sqrt{T}}{4L(\sqrt{\rho})^3}}$$

(b) i. Since $L, T, \rho > 0$, $\dfrac{df}{dL} < 0$, so f is decreasing as a function of L. This says that if L is decreased, f increases:



(ii) Since $L, T, \rho > 0$, $\dfrac{df}{dT} > 0$. So if T is increased, f increases.

(iii) Since $L, T, \rho > 0$, $\dfrac{df}{d\rho} < 0$. So if ρ is increased, f decreases.