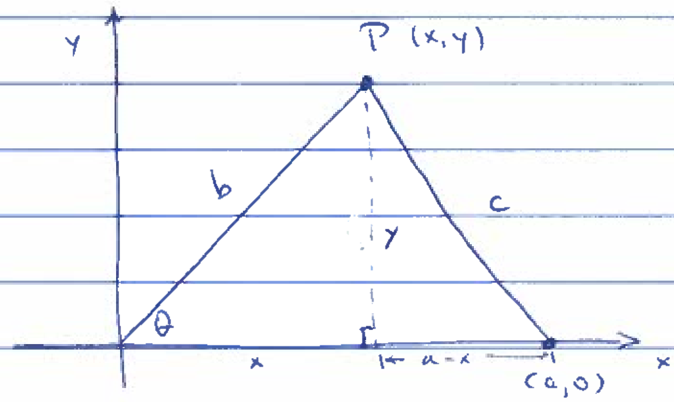
# MATH 135 – Problem Set 2 'B' Solutions

## Appendix C

41.

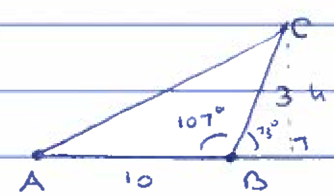



From the left triangle, $y = b\sin\theta$ and $x = b\cos\theta$.
Then in the triangle on the right, by the Pythagorean theorem:

$$c^2 = (a - b\cos\theta)^2 + (b\sin\theta)^2$$
$$= a^2 - 2ab\cos\theta + b^2\cos^2\theta + b^2\sin^2\theta$$
$$= a^2 - 2ab\cos\theta + b^2(\underbrace{\cos^2\theta + \sin^2\theta}_{1})$$
$$= a^2 - 2ab\cos\theta + b^2\cdot 1 \quad \text{(trig identity)}$$
$$\therefore\ c^2 = a^2 + b^2 - 2ab\cos\theta \quad \text{as desired.}$$

46. (a) Referring to the same diagram as above, the area of the triangle is $A = \frac{1}{2}(\text{base})(\text{height})$
$= \frac{1}{2}a \cdot b\sin\theta$.

(b) With the triangle




(not to scale ☺)

$$A = \tfrac{1}{2}\cdot 10\cdot 3\cdot \sin(107^\circ) \doteq 14.34\ \text{cm}^2$$

(The point of part (b) is to show the formula is still valid if $\theta$ is an obtuse angle. $h = 3\sin(73^\circ) = 3\sin(107^\circ)$.)

Section 1.7

40. <u>Solution 1</u>: ("algebraic" method)
When the line OC makes angle $\theta$ with the positive x-axis, C is the intersection of the lines

$$\begin{cases} y = 2a \\ y = (\tan\theta)\,x \end{cases}$$

So $2a = (\tan\theta)\,x$ and $x = \dfrac{2a}{\tan\theta} = 2a\cot\theta$.

C and P are on the same vertical line, so this is also the x-coordinate of P. To find the y-coordinate of P, we can note that it is the same as the y-coordinate of A.

A is the intersection of the circle $x^2+(y-a)^2 = a^2$ and the line $y = (\tan\theta)\,x$ with $x \neq 0$. Substituting, we get $x^2 + (\tan\theta\, x - a)^2 = a^2$, or

$$(1+\tan^2\theta)\,x^2 - 2a\tan\theta\, x = 0$$

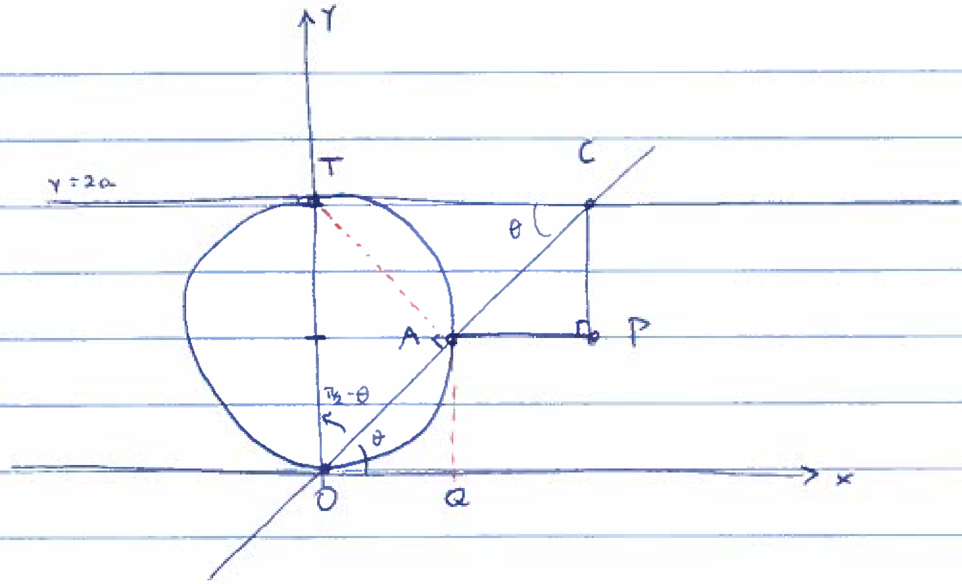
If $x \neq 0$, then $x = \dfrac{2a\tan\theta}{1+\tan^2\theta}$, and

$$y = (\tan\theta)\,x = \frac{2a\tan^2\theta}{1+\tan^2\theta} = \frac{2a\,\frac{\sin^2\theta}{\cos^2\theta}}{1+\frac{\sin^2\theta}{\cos^2\theta}} = \frac{2a\,\frac{\sin^2\theta}{\cos^2\theta}}{\frac{\cos^2\theta+\sin^2\theta}{\cos^2\theta}}$$

$$= 2a\sin^2\theta.$$

Hence $P = (x, y) = (2a\cot\theta,\ 2a\sin^2\theta)$. //

<u>Solution 2</u>: ("geometric" method) Consider the diagram as given in the problem:




In $\Delta OCT$, $\angle OCT = \theta$ (opposite angles formed by a transversal to two parallel lines). Therefore

$$\frac{|OT|}{|CT|} = \tan\theta, \quad \text{so } |CT| = 2a\cot\theta.$$

This is the x-coordinate of P. The y-coordinate of P is the same as $\boxed{|AQ| = |OA|\sin\theta}$. The triangle $\Delta OAT$ is inscribed in a semicircle, so

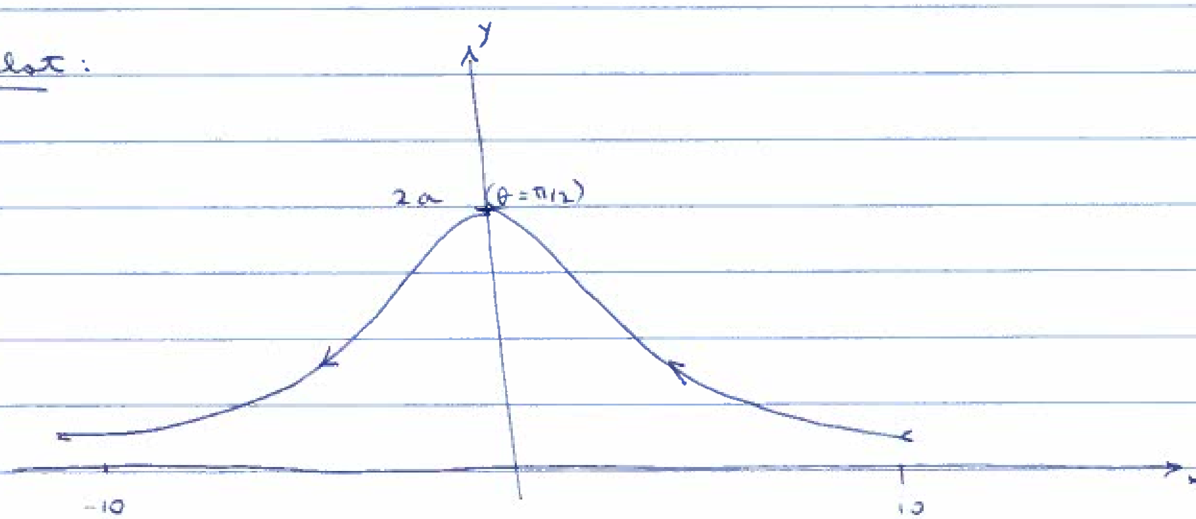
$\angle TAO$ is a right angle

it is also a right triangle. So

$$\boxed{|OA| = |OT|\cos\left(\tfrac{\pi}{2}-\theta\right) = 2a\left[\cos\left(\tfrac{\pi}{2}\right)\cos\theta + \sin\left(\tfrac{\pi}{2}\right)\sin\theta\right] = 2a\sin\theta}.$$

Combining the boxed equations gives $|AQ| = 2a\sin^2\theta$. This is the y-coordinate of P. $P = (2a\cot\theta,\ 2a\sin^2\theta)$.

The plot:



42.

(a) $y = (v_0 \sin\alpha)t - \frac{1}{2}gt^2$

$= 250t - 4.9t^2$

$= t(250 - 4.9t)$

this gives $y = 0$ if $t = 0$, or $t = \frac{250}{4.9} \doteq \boxed{51.02 \text{ sec}}$.

When $t = 51.02$, $x = 500 \cdot \frac{\sqrt{3}}{2} \cdot 51.02 = \boxed{22092 \text{ m}}$

The maximum height is attained when $t = \frac{51.02}{2}$ $\doteq 25.51$ sec. At that time, $y = (250)(25.51) - (4.9)(25.51)^2$ $\doteq \boxed{3189 \text{ m}}$.

(b) Plotting $x = \frac{500\sqrt{3}}{2}t$ $y = 250t - 4.9t^2$ in a large enough graphing window will confirm these answers. When the angle $\alpha$ is changed, we can solve for the distance as above:

$y = 500 \sin\alpha\, t - 4.9t^2 = 0$ when $t = 0$, $\frac{500 \sin\alpha}{4.9}$.

At $t = \frac{500 \sin\alpha}{4.9}$, $x = \frac{500^2}{4.9} \sin\alpha \cos\alpha = \frac{500^2}{9.8} \sin(2\alpha)$

by a trig identity you can find in Appendix C. For $0 < \alpha < \frac{\pi}{2}$, note $\sin(2\alpha)$ reaches a max when $\alpha = \frac{\pi}{4} = 45°$. So the distance traveled increases for $0 < \alpha < \frac{\pi}{4}$, but decreases for $\frac{\pi}{4} < \alpha < \frac{\pi}{2}$.

(c) From $x = (v_0 \cos\alpha)t$, we get $t = \frac{x}{v_0 \cos\alpha}$. Then in the y-equation: $y = (v_0 \sin\alpha) \cdot \frac{x}{v_0 \cos\alpha} - \frac{1}{2}g\left(\frac{x}{v_0 \cos\alpha}\right)^2$

$= (\tan\alpha)x - \frac{g}{2v_0^2 \cos^2\alpha}x^2$

Form is $y = Ax - Bx^2$, so path is a parabola.