MATH 131, section 1 – Calculus for Physical and Life Sciences
Solutions for Sample Exam Questions – Exam 3
November 19, 2013

I. Find y' and simplify.

(a) $y = \ln(x)\left(x^7 - \frac{4}{\sqrt{x}}\right)$

Solution: By the product rule,

$$\begin{aligned} y' &= \ln(x)(7x^6 + 2x^{-3/2}) + \frac{1}{x}\left(x^7 - \frac{4}{x^{1/2}}\right) \\ &= \ln(x)(7x^6 + 2x^{-3/2}) + x^6 - \frac{4}{x^{3/2}} \end{aligned}$$

(b) $y = (e^{2x} + 2)^3$.

Solution: By the chain rule (twice),

$$y' = 3(e^{2x} + 2)^2 \cdot e^{2x} \cdot 2 = 6e^{2x}(e^{2x} + 2)^2.$$

(c) $y = \frac{x+1}{3x^4-1}$.

Solution: By the quotient rule,

$$\begin{aligned} y' &= \frac{(3x^4 - 1)(1) - (x + 1)(12x^3)}{(3x^4 - 1)^2} \\ &= \frac{-9x^4 - 12x^3 - 1}{(3x^4 - 1)^2} \end{aligned}$$

(d) $y = \frac{\sin(x)}{1+\cos(x)}$

Solution: By the quotient rule,

$$\begin{aligned} y' &= \frac{(1 + \cos(x))(\cos(x)) - \sin(x)(-\sin(x))}{(1 + \cos(x))^2} \\ &= \frac{\cos(x) + \cos^2(x) + \sin^2(x)}{(1 + \cos(x))^2} \\ &= \frac{1}{1 + \cos(x)} \end{aligned}$$

(e) $y = \tan^{-1}(e^{5x})$.

Solution: By the derivative rule for inverse tangent and the chain rule,

$$y' = \frac{1}{1 + (e^{5x})^2} \cdot e^{5x} \cdot 5 = \frac{5e^{5x}}{1 + e^{10x}}.$$

(f) $xy^2 - 3y^3 + 2x^4 = 2$.

Solution: Since the equation involves both x and y we use *implicit differentiation*. Differentiating thinking of y as a function of x,

$$2xyy' + y^2 - 9y^2y' + 8x^3 = 0$$

Then solving for y':

$$y' = \frac{-y^2 - 8x^3}{2xy - 9y^2}.$$

(g) $y = \cos(x)^{x^3}$.

Solution: For functions of the form $u(x)^{v(x)}$, we use *logarithmic differentiation*. First take ln of both sides

$$\ln(y) = \ln(\cos(x)^{x^3}) = x^3 \ln(\cos(x)),$$

then differentiate using the product and chain rules:

$$\frac{1}{y}y' = -x^3 \tan(x) + 3x^2 \ln(\cos(x)).$$

So

$$y' = y(-x^3 \tan(x) + 3x^2 \ln(\cos(x))) = \cos(x)^{x^3}(-x^3 \tan(x) + 3x^2 \ln(\cos(x))).$$

II. The quantity of a reagent present in a chemical reaction is given by $Q(t) = t^3 - 3t^2 + t + 30$ grams at time t seconds for all $t \geq 0$.

(a) Over which intervals with $t \geq 0$ is the amount increasing? decreasing?

Solution: We need to determine the intervals where $Q'(t) = 3t^2 - 6t + 1$ is positive and negative. By the quadratic formula, $3t^2 - 6t + 1 = 0$ when

$$t = \frac{6 \pm \sqrt{24}}{6} = \frac{3 \pm \sqrt{6}}{3},$$

which are both positive numbers. By making a sign chart for Q' we see $Q'(t) > 0$ for t in $\left[0, \frac{3-\sqrt{6}}{3}\right) \cup \left(\frac{3+\sqrt{6}}{3}, +\infty\right)$. (Note the problem just said look at $t \geq 0$.) $Q'(t) < 0$ for t in $\left(\frac{3-\sqrt{6}}{3}, \frac{3+\sqrt{6}}{3}\right)$.

(b) Over which intervals is the rate of change of Q increasing? decreasing? The rate of change is increasing (decreasing) when $Q''(t) = 6t - 6$ is positive (negative). This is increasing for $t > 1$ and decreasing for $0 \leq t < 1$ (again, the problem said look only at $t \geq 0$ so we are ignoring $t < 0$).

III. A spherical balloon is being inflated at 20 cubic inches per minute. When the radius is 6 inches, at what rate is the radius of the balloon increasing? At what rate is the surface area increasing? (The volume of a sphere of radius r is $V = \frac{4\pi r^3}{3}$ and the surface area is $4\pi r^2$.)

Solution: This is a *related rates* problem. From the volume formula, since V and r are changing with time t,

$$\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}.$$

We are given that $\frac{dV}{dt} = 20$, so when $r = 6$:

$$20 = 4\pi(6)^2 \frac{dr}{dt},$$

so

$$\frac{dr}{dt} = \frac{20}{144\pi} = \frac{5}{36\pi}$$

(inches per minute). The surface area is changing at the rate

$$\frac{dA}{dt} = 8\pi r \frac{dr}{dt} = 8\pi \cdot 6 \cdot \frac{5}{36\pi} = \frac{20}{3}$$

(square inches per minute).

IV. All parts of this question refer to $f(x) = 4x^3 - x^4$.

(a) Find and classify all the critical numbers of f using the First Derivative Test.

Solution: The first derivative is $f'(x) = 12x^2 - 4x^3 = 4x^2(3 - x)$, so f has critical numbers at $x = 0$ and $x = 3$. The first derivative is positive for $x < 0$, positive for $0 < x < 3$, and negative for $x > 3$. Hence $x = 3$ is a local maximum, and $x = 0$ is neither a local max nor a local min.

(b) Over which intervals is the graph $y = f(x)$ concave up? concave down?

Solution: For concavity, we need $f''(x) = 24x - 12x^2 = 12x(2 - x)$. This is negative for $x < 0$ and $x > 2$, and positive for $0 < x < 2$. Hence $y = f(x)$ is concave up on $(0, 2)$ and concave down on $(-\infty, 0) \cup (2, +\infty)$. (Since the graph changes concavity at $x = 0, 2$, these are *points of inflection.*)

(c) Sketch the graph $y = f(x)$.

Solution: Omitted – Ask Gopal to sketch in the review session.

(d) Find the absolute maximum and minimum of $f(x)$ on the interval $[1, 4]$.

Solution: On this closed interval, we have a critical number at $x = 3$. The critical value is $f(3) = 4 \cdot 27 - 81 = 27$. The values at the endpoints are $f(1) = 4 - 1 = 3$ and $f(4) = 4 \cdot 64 - 256 = 0$. Hence the maximum value on the interval is $f(3) = 27$ and the minimum is $f(4) = 0$.

V. All three parts of this question refer to the function $f(x) = x^{2/3} - \frac{1}{5}x^{5/3}$.

(a) Find all the critical numbers of $f(x)$.

Solution: The derivative is

$$f'(x) = \frac{2}{3}x^{-1/3} - \frac{1}{3}x^{2/3} = \frac{1}{3}x^{-1/3}(2-x)$$

The function has critical numbers at $x = 0$ ($f'(0)$ is undefined – the graph has a cusp point there), and $x = 2$ ($f'(2) = 0$).

(b) Find all the inflection points of $f(x)$.

Solution: For this we need

$$f''(x) = \frac{-2}{9}x^{-4/3} - \frac{2}{9}x^{-1/3} = -\frac{2}{9}x^{-4/3}(1+x)$$

There is just one point of inflection at $x = -1$. Since $x^{-4/3} = \frac{1}{(x^{1/3})^4}$, that term is positive whenever it is defined, so the sign of f'' changes only at $x = -1$.

(c) For which of the critical numbers here is the Second Derivative Test applicable? Why? Determine the type of each such critical number using the Second Derivative Test.

Solution: The Second Derivative Test is applicable at critical numbers where the first derivative is equal to zero; here only at $x = 2$. We have $f''(2) = -\frac{2}{9} \cdot 2^{-4/3}(1+2) = -\frac{2}{3} \cdot 2^{-4/3} < 0$. Hence f has a *local maximum* at $x = 2$ since the graph is concave down there.