On Linear Stability in the N-Body Problem I, II and III

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2007 NCTS Workshop on Dynamical Systems
National Center for Theoretic Sciences
Hsinchu, Taiwan
May 14 - 18, 2007

Outline for Three Talks

1. The Planar *n*-Body Problem
2. Central Configurations and Relative Equilibria
3. Linear Stability of Relative Equilibria
4. Example: $1 + n$-gon
5. Equal Mass Case and Moeckel's Dominant Mass Conjecture
6. Example: Elliptic Lagrange Triangle Periodic Solutions
7. Symmetry in Hamiltonian Systems
8. Variational Methods
9. Example: Figure-eight Orbit
10. Other Orbits

Applications to other Areas

- Motion of n point vortices
- Mechanical systems involving a potential function dependent solely on mutual distances
- Hamiltonian systems
- Systems with integrals
- Systems with symmetric periodic orbits

The Planar n-Body Problem

$$\begin{aligned} m_i &= \text{mass of the } i\text{-th body} \\ \mathbf{q}_i &= \text{position of the } i\text{-th body in } \mathbb{R}^2 \\ \mathbf{p}_i &= m_i\, \dot{\mathbf{q}}_i \quad \text{(momentum)} \\ r_{ij} &= ||\mathbf{q}_i - \mathbf{q}_j|| \end{aligned}$$

$$\begin{aligned} \mathbf{q} &= (\mathbf{q}_1, \ldots, \mathbf{q}_n) \in \mathbb{R}^{2n} \\ \mathbf{p} &= (\mathbf{p}_1, \ldots, \mathbf{p}_n) \in \mathbb{R}^{2n} \\ M &= \text{diag}\,\{m_1, m_1, m_2, m_2, \ldots, m_n, m_n\} \end{aligned}$$

Newtonian potential function:

$$U(\mathbf{q}) = \sum_{i<j}^{n} \frac{m_i m_j}{r_{ij}}$$

Equations of motion:

$$
\begin{aligned}
m_i \ddot{\mathbf{q}}_i &= \frac{\partial U}{\partial \mathbf{q}_i}, \qquad i \in \{1, 2, \ldots n\} \\
&= \sum_{i \neq j}^{n} \frac{m_i m_j (\mathbf{q}_j - \mathbf{q}_i)}{r_{ij}^3}
\end{aligned}
$$

Hamiltonian system:

$$
\begin{aligned}
\dot{\mathbf{q}} &= M^{-1}\mathbf{p} = \frac{\partial H}{\partial \mathbf{p}} \\
\dot{\mathbf{p}} &= \nabla U(\mathbf{q}) = -\frac{\partial H}{\partial \mathbf{q}}
\end{aligned}
$$

$$
\begin{aligned}
H(\mathbf{q}, \mathbf{p}) &= K(\mathbf{p}) - U(\mathbf{q}) \\
K(\mathbf{p}) &= \sum_{i=1}^{n} \frac{||\mathbf{p}_i||^2}{2m_i} \qquad \text{Kinetic Energy}
\end{aligned}
$$

Equilibria?

- For $(\mathbf{q}, 0)$ to be an equilibrium point, $\nabla U(\mathbf{q}) = 0$.

$$U(\mathbf{q}) = \sum_{i<j}^{n} \frac{m_i m_j}{r_{ij}}$$

- But, U is a homogeneous potential of degree -1

$$\nabla U(\mathbf{q}) \cdot \mathbf{q} \;=\; -U(\mathbf{q}) \;<\; 0$$

- Therefore, there are no equilibrium points.
- Physically, this is to be expected.

Definition

The position vector $\mathbf{x}$ is a **central configuration (CC)** if there is some positive valued scalar function $r(t)$ such that

$$\mathbf{q}(t) = r(t)\,\mathbf{x}$$

is a solution to the n-body problem. The vector $\mathbf{x}$ may consist of collinear, planar or spatial positions.

Substituting this into the second order equation

$$M\ddot{\mathbf{q}} \;=\; \nabla U(\mathbf{q})$$

gives

$$M\,\ddot{r}\,\mathbf{x} = \nabla U(r\,\mathbf{x}) = r^{-2}\,\nabla U(\mathbf{x}). \tag{1}$$

Taking the dot product of (1) with $\mathbf{x}$ gives:

$$\ddot{r} = -\frac{\mu}{r^2}, \qquad \mu = \frac{U(\mathbf{x})}{\sum\; m_i||\mathbf{x}_i||^2}.$$

- Given a solution to the scalar ODE

$$\ddot{r} = -\frac{\mu}{r^2} \qquad \text{1d Kepler problem}$$

for some μ, the position vector $\mathbf{x}$ must satisfy

$$\nabla U(\mathbf{x}) + \mu M\mathbf{x} = 0. \tag{2}$$

- Thus, $\mathbf{x}$ is a *central configuration* if it satisfies equation (2) for some constant μ.

- While the ODE for $r(t)$ is straight-forward, the system (2) of nonlinear algebraic equations for $\mathbf{x}$ is extremely challenging!

- $r(t) = c\,t^{2/3}$ with $c^3 = 9\mu/2$ is a solution to the scalar ODE for $r(t)$ (homothetic solution arising from collision).

Periodic Solutions

Complexify and guess a solution of the form

$$\mathbf{q}(t) = \phi(t)\,\mathbf{x} \qquad \text{with } \phi(t) \in \mathbb{C}, \mathbf{x} \in \mathbb{C}^n$$

This leads to

$$\ddot{\phi} = -\frac{\mu|\phi|}{\phi^3} \qquad \text{2d Kepler problem}$$

and

$$\nabla U(\mathbf{x}) + \mu\, M\mathbf{x} = 0. \tag{3}$$

Given a planar CC **x** satisfying equation (3), attaching a particular solution of the Kepler problem to each body yields a solution to the full *n*-body problem.

- Rigid rotations (same shape and size)
- Elliptical periodic orbits (same shape, oscillating size)

Definition

A **relative equilibrium** for the n-body problem is a solution of the form

$$\mathbf{q}(t) = R(\omega t)\mathbf{x}$$

(a rigid rotation) where

$$R(t) = \begin{bmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{bmatrix}$$

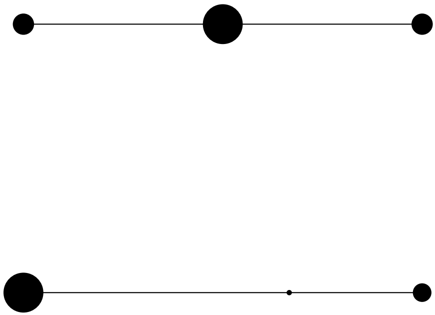
and

$$R(t)\mathbf{q} = (R(t)\mathbf{q}_1, R(t)\mathbf{q}_2, \ldots, R(t)\mathbf{q}_n).$$

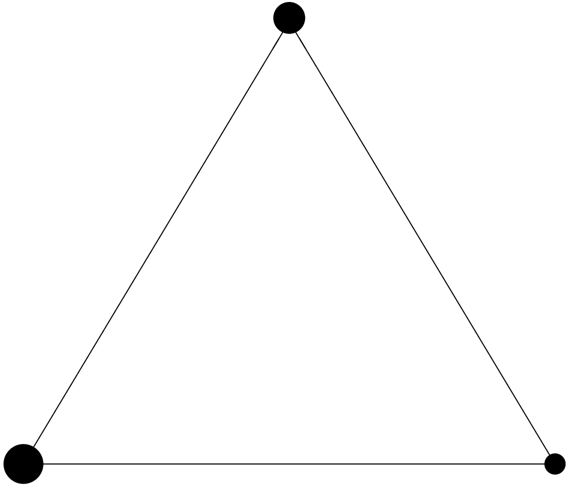
In order to have a relative equilibrium:

- $\mathbf{x}$ must be a planar cc, that is, $\nabla U(\mathbf{x}) + \mu\, M\mathbf{x} = 0$
- $\omega^2 = \mu = \frac{U(\mathbf{x})}{\sum m_i ||\mathbf{x}_i||^2}$ (rotation speed determined by $\mathbf{x}$)

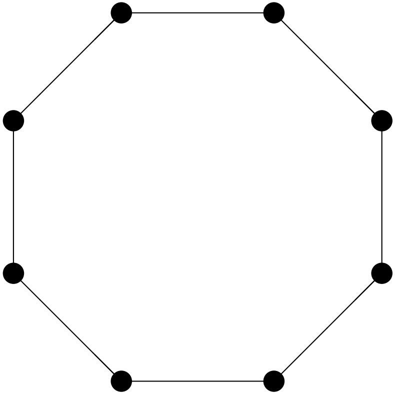
3-Body Collinear Configuration (Euler 1767)



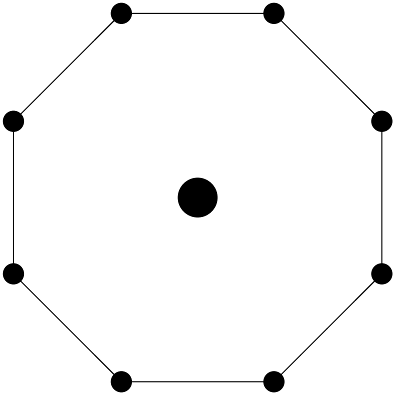
Equilateral Triangle (Lagrange 1772)



Regular n-gon (equal mass required for $n \geq 4$)



$1 + n$-gon (arbitrary central mass)



Used by Sir James Clerk Maxwell in 1859 in Stability of the Motion of Saturn's Rings (winner of the Adams Prize)

Rotating Coordinates

Let $J = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ and thus $e^{\omega J t} = \begin{bmatrix} \cos \omega t & \sin \omega t \\ -\sin \omega t & \cos \omega t \end{bmatrix}$

Changing to uniformly rotating coordinates with period $2\pi/\omega$ via

$$\mathbf{x}_i = e^{\omega J t}\mathbf{q}_i, \qquad \mathbf{y}_i = e^{\omega J t}\mathbf{p}_i$$

yields the autonomous Hamiltonian system

$$\begin{aligned} \dot{\mathbf{x}} &= \omega\mathbb{J}\mathbf{x} + M^{-1}\mathbf{y} &= \frac{\partial \hat{H}}{\partial \mathbf{y}} \\ \dot{\mathbf{y}} &= \nabla U(\mathbf{x}) + \omega\mathbb{J}\mathbf{y} &= -\frac{\partial \hat{H}}{\partial \mathbf{x}} \end{aligned}$$

where $\mathbb{J} = \operatorname{diag}\{J, J, \ldots, J\}$ and

$$\hat{H}(\mathbf{x}, \mathbf{y}) = K(\mathbf{y}) - U(\mathbf{x}) + \omega \mathbf{x}^{\mathrm{T}}\mathbb{J}\mathbf{y}.$$

- The new term added to the Hamiltonian, $\omega \mathbf{x}^{\mathrm{T}} \mathbb{J} \mathbf{y}$ is known as the Coriolis force.
- An equilibrium point in rotating coordinates $(\mathbf{x}, \mathbf{y})$ must satisfy

$$\begin{aligned} \mathbf{y} &= -\omega M \mathbb{J} \mathbf{x} \\ \nabla U(\mathbf{x}) &= \omega^2 \mathbb{J} M \mathbb{J} \mathbf{x} \quad = \quad -\omega^2 M \mathbf{x} \end{aligned} \tag{4}$$

- Equation (4) says that $\mathbf{x}$ is a planar cc (as expected)
- The values in the momentum vector $\mathbf{y}$ are the precise set of velocities that allow for an exact circular solution.

Degeneracies

$$\sum_{i \neq j} \frac{m_i m_j (\mathbf{x}_j - \mathbf{x}_i)}{r_{ij}^3} + \omega^2 \, m_i \mathbf{x}_i = 0 \tag{5}$$

$\mathbf{x} = (\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n)$ is a relative equilibrium implies that

$$c\mathbf{x} = (c\mathbf{x}_1, c\mathbf{x}_2, \ldots, c\mathbf{x}_n) \quad \text{and}$$

$$R\mathbf{x} = (R\mathbf{x}_1, R\mathbf{x}_2, \ldots, R\mathbf{x}_n)$$

are relative equilibria where c is a constant and $R \in SO(2)$.

The **moment of inertia** $I(\mathbf{x})$ is defined as

$$I(\mathbf{x}) = \frac{1}{2} \sum_{i=1}^{n} m_i \|\mathbf{x}_i\|^2$$

Equation (5) for a relative equilibrium can be viewed as a Lagrange multiplier problem: $(I(\mathbf{x}) = k)$

$$\nabla U(\mathbf{x}) + \omega^2 \, \nabla I(\mathbf{x}) = 0.$$

A Topological Viewpoint

Let S be the ellipsoid defined by $2I = 1$ (fixes scaling). Define an equivalence relation via $\mathbf{x} \sim R\mathbf{x}, \; R \in SO(2)$ (identify rotationally equivalent relative equilibria).

Critical points of $U([\mathbf{x}])$ on $S/\sim$ are relative equilibria.

Smale/Wintner/Chazy Question: For a given set of positive masses, is the number of relative equilibria equivalence classes finite? (Smale's 6th problem for the 21st century)

- $n = 3$ Euler, Lagrange
- $\frac{n!}{2}$ Collinear CC's Moulton
- 4 Equal masses Albouy (1995)
- $n = 4$ Hampton and Moeckel (2006)
- $n \geq 5$ Open problem

Linear Stability of Relative Equilibria

$$\begin{aligned}\dot{\mathbf{x}} &= \omega\mathbb{J}\mathbf{x} + M^{-1}\mathbf{y} \\ \dot{\mathbf{y}} &= \nabla U(\mathbf{x}) + \omega\mathbb{J}\mathbf{y}\end{aligned}$$

$$A = \begin{bmatrix} \omega\mathbb{J} & M^{-1} \\ D\nabla U(\mathbf{x}) & \omega\mathbb{J} \end{bmatrix}$$

Suppose that $\mathbf{v}$ is an eigenvector of A with eigenvalue λ, and write $\mathbf{v} = (\mathbf{v}_1, \mathbf{v}_2)$ with $\mathbf{v}_1, \mathbf{v}_2 \in \mathbb{C}^{2n}$. $A\mathbf{v} = \lambda\mathbf{v}$ becomes

$$\begin{aligned}\mathbf{v}_2 &= M(\lambda I - \omega\mathbb{J})\mathbf{v}_1 \\ B\mathbf{v}_1 &= 0\end{aligned}$$

where

$$B = M^{-1}D\nabla U(\mathbf{x}) + (\omega^2 - \lambda^2)I + 2\lambda\omega\mathbb{J}.$$

The characteristic polynomial for A is

$$P(\lambda) = \det\left[M^{-1}D\nabla U(\mathbf{x}) + (\omega^2 - \lambda^2)I + 2\lambda\omega\mathbb{J}\right].$$

Taking the transpose of B shows that $P(\lambda) = P(-\lambda)$. Thus, we require pure imaginary eigenvalues for linear stability.

Integrals of Motion

1. The subspace W_1 of $\mathbb{C}^{4n}$ spanned by the four vectors

$$(\mathbf{u}, 0),\ (0, M\mathbf{u}),\ (\mathbf{v}, 0),\ (0, M\mathbf{v})$$

   where $\mathbf{u} = (1, 0, 1, 0, \ldots)$ and $\mathbf{v} = (0, 1, 0, 1, \ldots)$ is invariant under A with eigenvalues $\pm i\omega, \pm i\omega$ corresponding to the the center of mass and total linear momentum integrals.

2. The subspace W_2 of $\mathbb{C}^{4n}$ spanned by the four vectors

$$(\mathbf{x}, 0),\ (0, M\mathbf{x}),\ (\mathbb{J}\mathbf{x}, 0),\ (0, \mathbb{J}M\mathbf{x})$$

   is invariant under A with eigenvalues $0, 0, \pm i\omega$ corresponding to the angular momentum integral and the fact that any relative equilibrium $\mathbf{x}$ is not isolated under scaling or rotation.

Definition

A relative equilibrium **x** is **non-degenerate** if the remaining $4n-8$ eigenvalues are nonzero. It is **spectrally stable** if the eigenvalues are pure imaginary and is **linearly stable** if in addition, the restriction of the matrix A to the skew-orthogonal complement of $W_1 \cup W_2$ is diagonalizable.

Skew-inner product: $\Omega(\mathbf{v}, \mathbf{w}) = \mathbf{v}^{\mathrm{T}} J \, \mathbf{w}$ $\quad J = \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix}$

Skew-orthogonal complement of W:

$$W^{\perp} \; = \; \{\mathbf{v} \in \mathbb{C}^{4n} : \Omega(\mathbf{v}, \mathbf{w}) = 0 \;\; \forall \mathbf{w} \in W\}$$

- W is an invariant subspace for a Hamiltonian matrix A iff $W^{\perp}$ is also invariant under A.
- Meyer and Schmidt (JDE 2005) show that it is possible to give a symplectic coordinate system that nicely decouples the above problem into three subsystems: W_1, W_2 and $(W_1 \cup W_2)^{\perp}$

Results on Linear Stability of Relative Equilibria

- Collinear (Andoyer 1906, M. G. Meyer 1933, Conley)
  Unstable for any choice of masses
- Equilateral Triangle (Gascheau 1843, Routh 1875)

$$27(m_1 m_2 + m_1 m_3 + m_2 m_3) < (m_1 + m_2 + m_3)^2$$

- Regular n-gon is unstable for all n (Moeckel 1995)
- Regular n-gon with a central mass m
  (Maxwell 1859, Moeckel 1994, Elmabsout 1994, GR 1997)
  Stable for $n \geq 7$ when $m > 0.435n^3$
- Rhombus is unstable (Ouyang, Xie 2006)
- Moeckel's dominant mass conjecture:
  For a relative equilibrium to be linearly stable, it must have a dominant mass.

Oscillation Speed

Suppose that $\lambda = \alpha + i\beta$ is an eigenvalue of a relative equilibrium with corresponding eigenvector $\mathbf{v} = (\mathbf{v}_1, \mathbf{v}_2)$.

$$\Omega(\mathbf{v}, \bar{\mathbf{v}}) = -2(i\beta\, \mathbf{v}_1^T M \bar{\mathbf{v}}_1 + \omega\, \mathbf{v}_1^T \mathbb{J} M \bar{\mathbf{v}}_1). \tag{6}$$

Let $\mathbf{v}_1 = (\mathbf{u}_1, \ldots, \mathbf{u}_n)$ with each component $\mathbf{u}_k \in \mathbb{C}^2$ written as $\mathbf{u}_k = (a_k + i\, b_k, c_k + i\, d_k)$. Formula (6) reduces to

$$\begin{aligned} -\frac{1}{2i}\Omega(\mathbf{v}, \bar{\mathbf{v}}) &= \left(\beta \sum_{k=1}^{n} m_k\, ||\mathbf{u}_k||^2 - i\omega \sum_{k=1}^{n} m_k\, \mathbf{u}_k^T J\, \bar{\mathbf{u}}_k\right) \\ &= \beta \sum_{k=1}^{n} m_k(a_k^2 + b_k^2 + c_k^2 + d_k^2) \\ &+ \omega \sum_{k=1}^{n} 2m_k(b_k c_k - a_k d_k). \end{aligned}$$

If $|\beta| > \omega$, then we can conclude that $\Omega(\mathbf{v}, \bar{\mathbf{v}}) \neq 0$. However, if $\alpha \neq 0$, then $\Omega(\mathbf{v}, \bar{\mathbf{v}}) = 0$ for any corresponding eigenvector $\mathbf{v}$.

Theorem

(GR 1999) Let A be the Hamiltonian matrix corresponding to a relative equilibrium $\mathbf{x}$ and suppose that A has an eigenvalue $\lambda = \alpha + i\beta$ with $\alpha \neq 0$. Then β satisfies $|\beta| \leq \omega$. Moreover, if $|\beta| = \omega$, then each component of the positional part of the corresponding eigenvector must be a complex multiple of the vector $[1, i]$.

Consequence: On the stable and unstable manifolds of a relative equilibrium, solutions oscillate at a slower rate than the periodic solution itself. A slight perturbation in an unstable direction forces the bodies to take longer to come back near their initial positions than in the unperturbed case.

A method for factoring $P(\lambda)$

Recall: The characteristic polynomial of a relative equilibrium $\mathbf{x}$ is

$$P(\lambda) = \det \left[M^{-1} D\nabla U(\mathbf{x}) + (\omega^2 - \lambda^2) I + 2\lambda\omega\mathbb{J} \right] \qquad (7)$$

The orthogonal complement of W with respect to M is

$$W^{\perp} = \{\mathbf{v} \in \mathbb{R}^{2n} : \mathbf{v}^{\mathrm{T}} M \mathbf{w} = 0 \ \ \forall \mathbf{w} \in W\}$$

Theorem

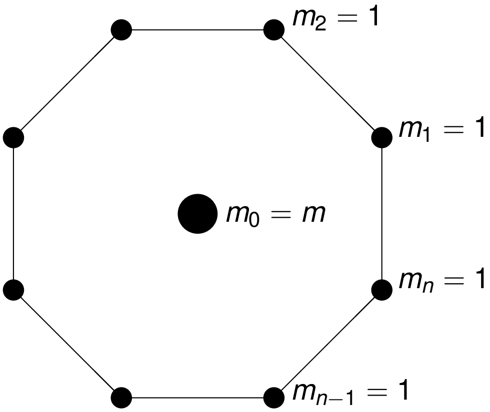
(Moeckel 1995) Suppose that $W \subset \mathbb{R}^{2n}$ is an invariant subspace for both $M^{-1} D\nabla U(\mathbf{x})$ and $\mathbb{J}$. Then $P(\lambda) = P_1(\lambda) P_2(\lambda)$, where P_1, P_2 are given by equation (7) with the matrices involved restricted to the subspaces W and $W^{\perp}$, respectively. Moreover, P_1 and P_2 are both even polynomials.

Could the Earth Have Rings?

Suppose that the moon was suddenly split into n equal pieces, with each “new” moon landing close to a vertex of a regular n-gon with the Earth at its center. Could such a configuration be stable?

NO, unless $n \in \{7, 8, \ldots, 13\}$.

The $1 + n$-gon Relative Equilibrium



Theorem

(GR '97) For $n \geq 7$, the $1+n$-gon relative equilibrium is linearly stable iff $m > h_n$. Moreover,

$$\lim_{n\to\infty} \frac{h_n}{n^3} = \frac{13+4\sqrt{10}}{2\pi^3} \cdot \sum_{k=1}^{\infty} \frac{1}{(2k-1)^3}$$

$$\approx 0.435036581297.$$

For n even,

$$h_n = 26A_n - 4B_n - \frac{\sigma_n}{2} + 4\sqrt{(5A_n - B_n)(8A_n - B_n)}$$

$$A_n = \sum_{k \text{ odd}}^{n-1} \frac{1}{8\sin^3(\pi k/n)} \qquad B_n = \sum_{k \text{ odd}}^{n-1} \frac{1}{2\sin(\pi k/n)}$$

$$\sigma_n = \sum_{k=1}^{n-1} \frac{1}{2\sin(\pi k/n)}$$

Rings around the Earth?

$$\frac{m_{\text{earth}}}{m_{\text{moon}}} \approx 81$$

Breaking the moon into n equal parts with $m_i = 1$ means that the central mass (Earth) is $m = 81n$.

However, for the stable case $n \geq 7$,

$$81n > 0.435n^3 \quad \text{only for} \quad 7 \leq n \leq 13$$

Thus, the Earth will not be heavy enough to sustain a stable ring for $n > 13$.

Finding Invariant Subspaces for the $1+n$-gon

Let $\theta_k = \dfrac{2\pi k}{n}$, $k \in \{1, \ldots, n\}$.

Then, $\mathbf{x} = ((0,0), \mathbf{x}_1, \ldots, \mathbf{x}_n)$ with $\mathbf{x}_k = (\cos\theta_k, \sin\theta_k)$, $k \in \{1, \ldots, n\}$ is a relative equilibrium with masses $m_k = 1\ \forall k \neq 0$ and $m_0 = m$.

Let $U_l = \mathrm{span}\{\mathbf{u}_l, \mathbb{J}\mathbf{u}_l\}$, where $\mathbf{u}_l = ((0,0), \mathbf{u}_{1l}, \ldots, \mathbf{u}_{nl})$ and $\mathbf{u}_{kl} = e^{i\theta_{kl}}\mathbf{x}_k$, $(\theta_{kl} = 2\pi kl/n)$.

For each $l \in \{2, 3, \ldots, [n/2]\}$, U_l is a two-dimensional complex invariant subspace for $M^{-1}D\nabla U(\mathbf{x})$.

However, when $l = 1$, this perturbation does not leave the central mass fixed at the origin. (This point was overlooked by Maxwell.) Instead of $(0,0)$, the first component of $\mathbf{u}_1$ must be $(-n/(2m), -in/(2m))$.

For $l \geq 2$, the restriction of the operator $M^{-1}D\nabla U(\mathbf{x})$ to the subspace U_l is

$$\begin{bmatrix} P_l - 3Q_l + 2m & -iR_l \\ iR_l & P_l + 3Q_l - m \end{bmatrix}$$

where

$$P_l = \sum_{k=1}^{n-1} \frac{1 - \cos\theta_k \cos\theta_{kl}}{2r_{nk}^3}, \qquad Q_l = \sum_{k=1}^{n-1} \frac{\cos\theta_k - \cos\theta_{kl}}{2r_{nk}^3} \quad \text{and}$$

$$R_l = \sum_{k=1}^{n-1} \frac{\sin\theta_k \sin\theta_{kl}}{2r_{nk}^3}.$$

When $l = 1$, we obtain the matrix

$$\begin{bmatrix} P_1 + 2m + n & -i(R_1 - n) \\ i(R_1 + n/2) & P_1 - m - n/2 \end{bmatrix}.$$

Outline of Proof

1. Take real and imaginary parts of $\mathbf{u}_l$ and $\mathbb{J}\mathbf{u}_l$ to derive subspaces of $\mathbb{R}^{2(n+1)}$ which are invariant under $M^{-1}D\nabla U(\mathbf{x})$ and $\mathbb{J}$.
2. Use these subspaces to factor $P(\lambda)$ completely into 4th and 8th degree polynomials. These polynomials are also even.
3. Derive necessary and sufficient conditions for linear stability based on the coefficients of these polynomials.
4. Examine how these conditions depend on the central mass m and locate a bifurcation value h_n where the $1+n$-gon becomes linearly stable.
5. Do asymptotics on h_n.

Note: It turns out that for n odd, the bifurcation value h_n is the root of a *fifth* degree polynomial in m. However, we are able to estimate h_n by bounding it below and above by quantities which are asymptotic to the same thing.

Some Concluding Remarks

1. The $n \geq 7$ requirement comes from the perturbation for $l = 1$ which moves the central mass. For $3 \leq n \leq 6$, no matter how large the central mass, the eigenvalues from this invariant subspace are always off the imaginary axis.
2. As l increases, the invariant subspaces correspond to perturbations with more and more twisting of the ring, thus requiring a larger central mass for stability.
3. In the case n is even, the final perturbation ($l = n/2$) before achieving stability alternates between pushing bodies directly toward or away from the central mass. Remarkably, Maxwell guessed this invariant subspace and calculated τ to be 0.4352 in 1859.
4. Calculating linear stability of relative equilibria analytically is a hard but useful problem!

Outline for Second Talk

1. Equal Mass Case and Moeckel's Dominant Mass Conjecture
2. Example: Elliptic Lagrange Triangle Periodic Solutions
   (This last topic was covered with hand-written transparencies.)

Definition

A **relative equilibrium** for the n-body problem is a solution of the form

$$\mathbf{q}(t) = R(\omega t)\mathbf{x}$$

(a rigid rotation) where

$$R(t) = \begin{bmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{bmatrix}$$

and

$$R(t)\mathbf{q} = (R(t)\mathbf{q}_1, R(t)\mathbf{q}_2, \ldots, R(t)\mathbf{q}_n).$$

In order to have a relative equilibrium:

- $\mathbf{x}$ must be a planar cc, that is, $\nabla U(\mathbf{x}) + \mu M\mathbf{x} = 0$
- $\omega^2 = \mu = \frac{U(\mathbf{x})}{2I(\mathbf{x})}$ where $I(\mathbf{x}) = \frac{1}{2}\sum m_i||\mathbf{x}_i||^2$

Linear Stability of Relative Equilibria

Let $J = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$, $\quad \mathbb{J} = \text{diag}\,\{J, J, \ldots, J\}$ and

$$B = M^{-1} D\nabla U(\mathbf{x}) + (\omega^2 - \lambda^2) I + 2\lambda\omega\mathbb{J}.$$

- The characteristic polynomial for a relative equilibrium is $P(\lambda) = \det(B)$.
- $P(\lambda)$ is even. Thus we require the eigenvalues to be purely imaginary to have linear stability.
- 8 eigenvalues always appear: $0, 0, \pm i\omega, \pm i\omega, \pm i\omega$ arising from the integrals of the planar n-body problem. Linear stability is determined by the remaining $4n - 8$ eigenvalues.
- The only known examples of linearly stable relative equilibria (eg. equilateral triangle, $1 + n$-gon) contain a dominant mass.

Goal: Show that equal mass relative equilibria are always unstable.

The 2×2 diagonal blocks of B are given by

$$\begin{bmatrix} d_{i\,i} + \omega^2 - \lambda^2 & d_{i\,i+1} + 2\lambda\omega \\ d_{i+1\,i} - 2\lambda\omega & d_{i+1\,i+1} + \omega^2 - \lambda^2 \end{bmatrix}$$

where d_{ij} is the ij-th entry of $M^{-1} D\nabla U(\mathbf{x})$.

Therefore $P(\lambda) = \lambda^{4n} + (2n\omega^2 - \mathrm{tr}(M^{-1} D\nabla U(\mathbf{x}))\,\lambda^{4n-2} + \cdots$ which implies

$$\frac{1}{2}\sum_{i=1}^{4n} \lambda_i^2 \;=\; \mathrm{tr}(M^{-1} D\nabla U(\mathbf{x})) - 2n\omega^2$$

where λ_i is an eigenvalue of A.

Direct calculation reveals:

$$\mathrm{tr}(M^{-1} D\nabla U(\mathbf{x})) \;=\; \sum_{i<j}^{n} \frac{m_i + m_j}{r_{ij}^3}\,.$$

Theorem

(GR 1999) One-half of the sum of the squares of the eigenvalues of a relative equilibrium $\mathbf{x}$ *with rotation speed* $\omega = U(\mathbf{x})/2I(\mathbf{x})$ *is given by*

$$\sum_{i<j} \frac{m_i + m_j}{r_{ij}^3} \quad - 2n\,\omega^2$$

Corollary

(GR 1999) A necessary condition for a relative equilibrium $\mathbf{x}$ *to be spectrally stable is*

$$\sum_{i<j} \frac{m_i + m_j}{r_{ij}^3} \quad < (2n-3)\,\frac{U(\mathbf{x})}{2I(\mathbf{x})}.$$

Example: This inequality is not satisfied by the regular n-gon relative equilibrium for $n \geq 7$ (quick proof of instability).

Fix the scaling of a relative equilibrium $\mathbf{x}$ so that

$$2I(\mathbf{x}) \ = \ \sum \ m_i||\mathbf{x}_i||^2 \ = \ 1 \qquad \text{(mass ellipsoid)}$$

A necessary condition for stability is then

$$\sum_{i<j} \frac{m_i + m_j}{r_{ij}^3} \ < \ (2n-3) \sum_{i<j} \frac{m_i m_j}{r_{ij}}. \tag{8}$$

Problem: The more bodies we consider, the closer they become on the mass ellipsoid. The left-hand side of (8) begins to dominate the right-hand side, leading to instability.

Theorem

(GR 1999) Any relative equilibrium of n equal masses is not spectrally stable for $n \geq 24,306$*.*

Proof Outline:

Set $m_i = 1$ for each i:

$$\sum_{i<j} \frac{1}{r_{ij}^3} \; < \; \frac{n}{2I} \sum_{i<j} \frac{1}{r_{ij}}$$

This motivates setting $2I = c\,n$, (with $c < 1$ to be chosen optimally)

$$\sum_{i=1}^{n} \|\mathbf{x}_i\|^2 = c\,n$$

giving a bound on the distance bodies can be from the origin. In fact, $[n(1-c)]$ bodies ($[\star]$ = greatest integer) are guaranteed to be inside the unit disk D.
To prove instability, it suffices to show

$$\sum_{i<j} \frac{1}{r_{ij}^3} - \frac{1}{c\,r_{ij}} \; > 0.$$

Let $f_c(r) = \frac{1}{r^3} - \frac{1}{cr}$. Let K_ϵ be the number of regions of diameter ϵ needed to cover the unit disk D with $\epsilon < \sqrt{c}$. Let

$$N_{c,\epsilon} = \left[\frac{[n(1-c)]}{K_\epsilon} \right]$$

be the approximate number of bodies in each ϵ-region (if they were to be evenly distributed). For ϵ sufficiently small, these mutual distances contribute positively to our instability inequality.

Let P_c be a lower bound on the number of mutual distances $< \sqrt{c}$.

$$\begin{aligned} \sum_{i<j} \frac{1}{r_{ij}^3} - \frac{1}{c} \cdot \frac{1}{r_{ij}} &> f_c(\epsilon) \cdot K_\epsilon \binom{N_{c,\epsilon}}{2} - \gamma_c \left(\binom{n}{2} - P_c \right) \\ &= A_{c,\epsilon}\, n^2 + B_{c,\epsilon}\, n \end{aligned} \tag{9}$$

- Choose ϵ small enough and c large enough to make sure that the coefficient in front of n^2 is positive and the positive root $n = \hat{n}$ of the quadratic (9) is as small as possible ($\epsilon = 0.029255$, $c = 0.370093$). This gives $\hat{n} \approx 26,000$.
- Further fine tuning (accounting for leftover mutual distances from an equal distribution) gets the minimum possible value of n down to $24,306$.
- This is clearly not the optimal result, as it is expected that **none** of the equal mass relative equilibria are spectrally stable. However, the stability inequality is indeed satisfied for certain equal mass configurations for small n even though these relative equilibria are unstable. (ie. our condition is necessary but not sufficient)
- One can use the same arguments to show that the collinear equal mass relative equilibria are unstable for $n \geq 22$.

Outline for Third Talk

1. Symmetry in Hamiltonian Systems
2. Variational Methods
3. Example: Figure-eight Orbit
4. Other Orbits

Some Quotes

- *You can go a long way just knowing Linear Algebra and Calculus.*
  - Rick Moeckel, speaking to grad students in his Introduction to Classical Mechanics class, April 5, 2004.

- *How the heck can that thing be stable?!*
  - Jim Walsh, November 2003.

- *All the choreographies found [in the n-body problem],* **except the eight**, *are unstable.*
  - Carles Simó in New families of Solutions in *N*-Body Problems, Proceedings of the ECM 2000, Barcelona (July, 10-14).

Variational Methods

Goal: Find special planar periodic solutions using Hamilton's principle of least action.

Let $\Sigma = \{\mathbf{q} \in \mathbb{R}^{2n} : \mathbf{q}_i = \mathbf{q}_j \text{ for some } i \neq j\}$ (collision set). The configuration space for the n-body problem is $\mathbb{R}^{2n} - \Sigma$. Let Γ_T denote the space of all absolutely continuous loops of period T in $\mathbb{R}^{2n} - \Sigma$.

The **action** of a path $\gamma \in \Gamma_T$ is

$$A(\gamma) = \int_0^T K(\dot{\gamma}(t)) + U(\gamma(t))\, dt$$

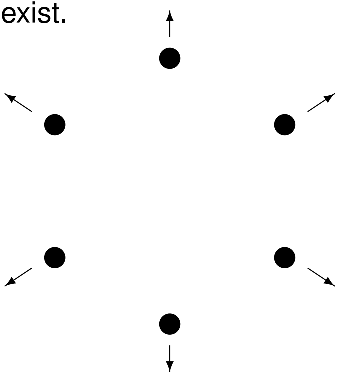
For the problem to be feasible, we need to restrict our paths to special classes, eg. by imposing symmetry or homotopical constraints.

Since $K \geq 0$ and $U > 0$, $A(\gamma) > 0$. Moreover, as $\gamma \to \Sigma$, $U(\gamma) \to \infty$. Therefore, we typically seek to **minimize** the action.

Problems Using Variational Methods

The configuration space $\mathbb{R}^{2n} - \Sigma$ is not compact.

1. Minima might not exist.



2. A minimizing trajectory may contain collisions. This occurs in the Kepler problem where the minimizing solution is independent of the eccentricity. The ejection/collision solution is an action minimizer (Gordon).

Theorem

(Chenciner, Montgomery 2000) There exists a figure-eight shaped curve $\mathbf{q} : (\mathbb{R}/T\mathbb{Z}) \mapsto \mathbb{R}^2$ *such that*

1. $\mathbf{q}(t) + \mathbf{q}(t + T/3) + \mathbf{q}(t + 2T/3) = 0 \ \ \forall t$
   (center of mass is at the origin.)
2. *Symmetry*

$$\mathbf{q}(t + T/2) = -\bar{\mathbf{q}}(t), \ \mathbf{q}(-t + T/2) = \bar{\mathbf{q}}(t) \ \ \forall t$$

3. $(\mathbf{q}(t + 2T/3), \mathbf{q}(t + T/3), \mathbf{q}(t))$ *is a zero angular momentum, periodic solution to the planar 3-body problem with equal masses.*

A solution where the n bodies follow each other along a single closed curve with equal phase shift is called a **choreography.**

Proof Outline:

Construct the orbit on the shape sphere, the space of oriented triangles.

1. Search for minimizers over the class of paths Γ traveling from an Euler central configuration (with say $\mathbf{q}_3$ at the center) to an isosceles configuration (with say $r_{12} = r_{13}$). This path will be $1/12$th of the full periodic orbit.
2. Let A_c be the smallest possible action for a path with collisions in Γ (compute via Kepler problem). Choose a simple test path (constant speed and potential) and compute its action A (numerically). Collisions are excluded by showing $A < A_c$.
3. The boundary terms of the first variation (integration by parts) and the symmetries induced by equality of masses allows for eleven copies of the minimizer to be fit together to create the full orbit.
4. A special area formula is used to reconstruct the motion in the phase space. By showing that the angular momentum of any one of the bodies vanishes only as the body passes through the origin, this implies the curve is a figure eight.

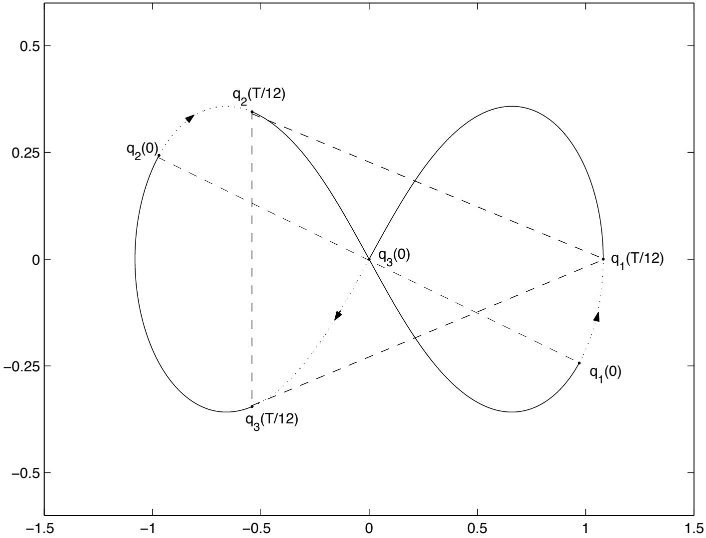


Figure: The first 12th of the figure-eight orbit (dotted), traveling from an Euler collinear central configuration to an isosceles triangle.

Why is the linear stability of the figure-eight so surprising?

- The regular n-gon circular choreographies (equal mass) are all unstable.
- The Lagrange equilateral triangle solution is linearly stable only when one mass dominates the others.
- If $n \geq 24,306$, all equal mass relative equilibria are unstable.
- The $1 + n$-gon relative equilibrium is linearly stable iff the central mass is at least $0.435n^3$.
- All other known choreographies appear to be unstable.

On the other hand: Adding eccentricity to an unstable relative equilibrium could make it linearly stable (eg. Lagrange equilateral triangle)

Linear Stability of Periodic Orbits

Set $J = \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix}$. Suppose $\zeta(t)$ is a T-periodic solution to the Hamiltonian system $\dot{z} = J\nabla H(z)$. The associated **linear system** is

$$\dot{\xi} = JD^2 H(\zeta(t))\,\xi, \qquad \xi(0) = I$$

The **fundamental matrix** solution $X(t)$ satisfies $X(t+T) = X(t)\,X(T)$.

$X(T)$ is the **monodromy matrix**, measuring the non-periodicity of solutions to the linearized equations. Its eigenvalues, the **characteristic multipliers**, determine the stability of the periodic solution.

$X(T)$ is a symplectic matrix with multipliers symmetric with respect to the unit circle. Linear stability requires all of the multipliers to be on the unit circle.

Recall: Each integral yields a multiplier of $+1$. For example, the monodromy matrix has two 2×2 Jordan blocks of the form

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

arising from the integrals due to the center of mass and total linear momentum.

- Center of mass, total linear momentum (4)
- SO(2) symmetry, angular momentum (2)
- Hamiltonian (1)
- Periodic orbit (1)

Definition: A periodic solution of the planar n-body problem has 8 trivial characteristic multipliers of $+1$. The solution is **spectrally stable** if the remaining multipliers lie on the unit circle and **linearly stable** if in addition, $X(T)$ restricted to the reduced space is diagonalizable.

Reductions Using Symmetry

Lemma

(Time-forward Symmetry) *Suppose that $\gamma(t)$ is a a symmetric T-periodic solution of a Hamiltonian system with Hamiltonian H and symmetry matrix S such that*

1. *for some $N \in \mathbb{N}$, $\gamma(t + T/N) = S\gamma(t) \;\; \forall t$*
2. *$H(Sx) = H(x)$*
3. *$SJ = JS$*
4. *S is orthogonal.*

Then the fundamental matrix solution $X(t)$ to the linearization problem $\dot{\xi} = JD^2H(\gamma(t))\,\xi,\; \xi(0) = I$ satisfies

$$X(t + \tfrac{T}{N}) = SX(t)S^{\mathrm{T}}X(\tfrac{T}{N})$$

and

$$X(\tfrac{kT}{N}) = S^k(S^{\mathrm{T}}X(\tfrac{T}{N}))^k \qquad \forall\, k \in \mathbb{N}.$$

Lemma

(Time-reversing Symmetry) *Suppose that $\gamma(t)$ is a T-periodic solution of a Hamiltonian system with Hamiltonian H and time-reversing symmetry S such that*

1. *for some $N \in \mathbb{N}$, $\gamma(-t + T/N) = S\gamma(t)\ \forall t$*
2. $H(Sx) = H(x)$
3. $SJ = -JS$
4. *S is orthogonal.*

Then the fundamental matrix solution $X(t)$ to the linearization problem $\dot{\xi} = JD^2H(\gamma(t))\,\xi, \quad \xi(0) = I$ satisfies

$$X(-t + \tfrac{T}{N}) = SX(t)S^{\mathrm{T}}X(\tfrac{T}{N}).$$

and

$$X(\tfrac{T}{N}) = SB^{-1}S^{\mathrm{T}}B \qquad \text{where } B = X(\tfrac{T}{2N}).$$

Definition

A **choreography** is a planar T-periodic solution of the n-body problem where all bodies follow the same loop $q(t)$ with equal time spacing. $\gamma(t) = (q(t + \frac{n-1}{n}T), q(t + \frac{n-2}{n}T), \cdots, q(t + \frac{T}{n}), q(t))$ is a solution, where $q(t + T) = q(t)$.

Let σ be the $2n \times 2n$ permutation matrix determined by

$$\sigma(q_1, q_2, \cdots, q_n)^{\mathsf{T}} = (q_n, q_1, q_2, \cdots, q_{n-1})^{\mathsf{T}}$$

It follows that:

- $\gamma(t + T/n) = \sigma\gamma(t)\ \forall t$
- Assuming equal masses, $H(\sigma q, \sigma \dot{q}) = H(q, \dot{q})$.
- Let $P = \begin{bmatrix} \sigma & 0 \\ 0 & \sigma \end{bmatrix}$. P is orthogonal, symplectic, commutes with J and $P^n = I$.

Theorem

(GR 2007) Suppose that $\gamma(t)$ is a choreography and let $X(t)$ be the fundamental matrix solution to the linearized equations about $\gamma(t)$. Then the monodromy matrix for γ is

$$(P^T X(\tfrac{T}{n}))^n \quad \textit{where} \quad P = \begin{bmatrix} \sigma & 0 \\ 0 & \sigma \end{bmatrix}.$$

Lemma

Let A be the fundamental matrix solution evaluated over the first piece of the orbit, $A = X(T/n)$. Then

$$X(\tfrac{kT}{n}) = P^k (P^T A)^k$$

Note: Since we want the eigenvalues to be on the unit circle, the linear stability analysis reduces to studying the symplectic matrix $P^{\mathrm{T}} X(\frac{T}{n})$.

Jacobi Coordinates

Set $m_i = 1 \ \forall i$.
Recall: $\mathbf{q}_i$ = position, $\mathbf{p}_i = \dot{\mathbf{q}}_i$ = momentum

$$
\begin{aligned}
\mathbf{u}_1 &= \frac{1}{\sqrt{2}}(\mathbf{q}_3 - \mathbf{q}_2) \\
\mathbf{u}_2 &= \sqrt{\frac{2}{3}}(\mathbf{q}_1 - \frac{1}{2}(\mathbf{q}_2 + \mathbf{q}_3)) \\
\mathbf{u}_3 &= \frac{1}{3}(\mathbf{q}_1 + \mathbf{q}_2 + \mathbf{q}_3)
\end{aligned}
\qquad
\begin{aligned}
\mathbf{v}_1 &= \frac{1}{\sqrt{2}}(\mathbf{p}_3 - \mathbf{p}_2) \\
\mathbf{v}_2 &= \sqrt{\frac{2}{3}}(\mathbf{p}_1 - \frac{1}{2}(\mathbf{p}_2 + \mathbf{p}_3)) \\
\mathbf{v}_3 &= \mathbf{p}_1 + \mathbf{p}_2 + \mathbf{p}_3.
\end{aligned}
$$

Set $\mathbf{u}_3 = \mathbf{v}_3 = 0$. Inertia becomes

$$I = ||\mathbf{u}_1||^2 + ||\mathbf{u}_2||^2$$

New Hamiltonian:

$$H(\mathbf{u}, \mathbf{v}) = \frac{1}{2}(||\mathbf{v}_1||^2 + ||\mathbf{v}_2||^2) - \frac{1}{r_{12}} - \frac{1}{r_{13}} - \frac{1}{r_{23}}$$

Shape Sphere Coordinates

Use the Hopf map and choose the conjugate coordinates carefully:

$$
\begin{aligned}
w_1 &= ||\mathbf{u}_1||^2 - ||\mathbf{u}_2||^2 & z_1 &= \tfrac{1}{2I}(\mathbf{u}_1 \cdot \mathbf{v}_1 - \mathbf{u}_2 \cdot \mathbf{v}_2) \\
w_2 &= 2(\mathbf{u}_1 \cdot \mathbf{u}_2) & z_2 &= \tfrac{1}{2I}(\alpha\, \mathbf{u}_1 \cdot \mathbf{v}_1 - \beta\, \mathbf{u}_2 \times \mathbf{v}_2 + \mathbf{u}_1 \cdot \\
w_3 &= 2(\mathbf{u}_1 \times \mathbf{u}_2) & z_3 &= \tfrac{1}{2I}(\beta\, \mathbf{u}_1 \cdot \mathbf{v}_1 + \alpha\, \mathbf{u}_2 \times \mathbf{v}_2 + \mathbf{u}_1 \times \\
w_4 &= \arg(\mathbf{u}_1) & z_4 &= \mathbf{u}_1 \times \mathbf{v}_1 + \mathbf{u}_2 \times \mathbf{v}_2
\end{aligned}
$$

where $\alpha = (\mathbf{u}_1 \cdot \mathbf{u}_2)/||\mathbf{u}_1||^2$ and $\beta = (\mathbf{u}_1 \times \mathbf{u}_2)/||\mathbf{u}_1||^2$.

z_4 is angular momentum; set $z_4 = c$.

$$H(w, z) = K(z)I(w) - U(w) + \frac{c}{I + w_1}(c + 2w_3 z_2 - 2w_2 z_3)$$

where

$$K = \tfrac{1}{2}(z_1^2 + z_2^2 + z_3^2) \quad \text{and} \quad I = (w_1^2 + w_2^2 + w_3^2)^{1/2}$$

$$U = \frac{1}{\sqrt{I + w_1}} + \frac{1}{\sqrt{I - \frac{1}{2}w_1 + \frac{\sqrt{3}}{2}w_2}} + \frac{1}{\sqrt{I - \frac{1}{2}w_1 - \frac{\sqrt{3}}{2}w_2}}$$

- Fixing $I = 1$ gives the shape sphere.
- Collinear configurations correspond to $w_3 = 0$.
- Isosceles triangles with q_1 at the apex correspond to $w_2 = 0$.
- Equilateral triangles are $(0, 0, \pm 1)$.
- Lagrange relative equilibrium: $(a \neq 0)$

$$w = (0, 0, a, (c/a)t) \quad z = (0, -c/(2a), 0, c)$$

where $c^2 = 3\sqrt{a}$.

Instability of Lagrange Equilateral Triangle Solution

Set $\omega = c/\alpha$. Then $w_4(t) = \omega t$ and the period of the orbit is $2\pi/\omega$.
$\dot{\xi} = JD^2H(\Delta)\xi$ is no longer time dependent.
Compute

$$JD^2H(\Delta) = \begin{bmatrix} 0 & 0 & 0 & 4\alpha & 0 & 0 \\ -2\omega & 0 & -2\omega & 0 & 4\alpha & 0 \\ 0 & -2\omega & 0 & 0 & 0 & 4\alpha \\ -\frac{5\omega^2}{8\alpha} & 0 & -\frac{\omega^2}{\alpha} & 0 & 2\omega & 0 \\ 0 & -\frac{5\omega^2}{8\alpha} & 0 & 0 & 0 & 2\omega \\ -\frac{\omega^2}{\alpha} & 0 & -\frac{5\omega^2}{4\alpha} & 0 & 2\omega & 0 \end{bmatrix}$$

Symmetry: Let $S = \text{diag}\{1, -1, 1, -1, 1, -1\}$. S fixes the Lagrange solution, leaves H unchanged and $SJ = -JS$. It follows that:

$$JD^2H(\Delta) = -SJD^2H(\Delta)S$$

Char. multipliers come from $\exp(JD^2H(\Delta)2\pi/\omega)$:
$1, 1, e^{\sqrt{2}\pi}, e^{\sqrt{2}\pi}, e^{-\sqrt{2}\pi}, e^{-\sqrt{2}\pi}$

Stability Reductions for the Figure-Eight

Let $X(t)$ be the fundamental matrix solution for the linearized equations about the figure-eight.

1. Permutation symmetry (choreography or 240° rotation on shape sphere) to reduce monodromy matrix to $(P^T X(\frac{T}{3}))^3$
2. 120° rotation and reflection about equator to reduce to $(P_1^T X(\frac{T}{6}))^6$
3. Reflection about isosceles meridian M_1 and time-reversal symmetry gives
$$(P_1^T S X^{-1}(\tfrac{T}{12}) S X(\tfrac{T}{12}))^6$$

Letting $Q = P_1^T S$ and $C = X(\frac{T}{12})$ gives the monodromy matrix in factored form:

$$(QC^{-1}SC)^6$$

Theorem

(GR 2007) The monodromy matrix for the figure-eight is $(QC^{-1}SC)^6$ *where* $C = X(T/12)$,

$$Q = \begin{bmatrix} R & 0 \\ 0 & -R \end{bmatrix}, \quad R = \begin{bmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} & 0 \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

and

$$S = diag\,\{1, -1, 1, -1, 1, -1\}.$$

Recall: $S = \text{diag}\,\{1,-1,1,-1,1,-1\}$. Let $D = C^{-1}SC$.

$$X(T) = (QC^{-1}SC)^6 = (QD)^6$$

Stability determined by eigenvalues of QD.

- Eigenvalues of both Q and D are $\pm 1, \pm 1, \pm 1$.
- Q and D are symplectic with multiplier -1.
- Q is orthogonal and symmetric.
- Both Q and D are involutions: $Q^2 = D^2 = I$.
- D has special form

$$D = \begin{bmatrix} a & b_1 \\ -b_2 & -a^{\mathsf{T}} \end{bmatrix}$$

where b_1, b_2 are symmetric.

Choosing a Good Basis

Let $\zeta(t)$ represent the figure-eight on the reduced space. Since $\dot{\zeta} = J\nabla H(\zeta)$, it follows that $\ddot{\zeta} = JD^2 H(\zeta(t))\dot{\zeta}$.
Therefore, $\dot{\zeta}(T) = \dot{\zeta}(0)$ is an evec. with eval. $+1$ for both the monodromy matrix and QD.

Lemma

Let $Y(t)$ be the fundamental matrix solution to the linearized equations for the figure-eight with arbitrary initial conditions $Y(0) = Y_0$. The monodromy matrix is similar to

$$(Y_0^{-1} Q Y_0 \, C^{-1} S C)^6$$

where $C = Y(T/12)$.

Choose a very special Y_0 to be orthogonal, symplectic and containing the eigenvectors of Q. Moreover, take the fourth column of Y_0 to be $\dot{\zeta}(0)$.

With this choice of Y_0, $Y_0^{-1} Q Y_0 = \Lambda = \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix}$ and the fourth column of C is

$$\dot{\zeta}(T/12) = (0, *, 0, *, 0, *)^{\mathrm{T}}$$

The matrix determining stability is then ΛD with the special property that

$$\tfrac{1}{2}(\Lambda D + D\Lambda) = \begin{bmatrix} a & 0 \\ 0 & a^{\mathrm{T}} \end{bmatrix} \qquad \text{very nice!}$$

If $(\Lambda D)v = \lambda v$, then $(D\Lambda)v = \frac{1}{\lambda} v$ and

$$(\Lambda D + D\Lambda)v = (\lambda + \tfrac{1}{\lambda})v$$

For the figure-eight to be stable, the eigenvalues of a must be real and between -1 and 1.
By construction, a^{T} has the form

$$\begin{bmatrix} 1 & * & * \\ 0 & u_2 \cdot SJu_5 & u_2 \cdot SJu_6 \\ 0 & u_3 \cdot SJu_5 & u_3 \cdot SJu_6 \end{bmatrix}$$

where u_i is the i-th column of $C = Y(T/12)$.

Theorem

Let u_i be the i-th column of $C = Y(T/12)$, where $Y(t)$ is the fundamental matrix solution with special initial condition Y_0. The figure-eight is spectrally stable if and only if the eigenvalues λ_i of

$$\begin{bmatrix} u_2 \cdot SJu_5 & u_2 \cdot SJu_6 \\ u_3 \cdot SJu_5 & u_3 \cdot SJu_6 \end{bmatrix}$$

are real and satisfy $-1 \leq \lambda_i \leq 1$.

Numerical calculations via MATLAB show $\lambda_1 = 0.2098\ldots$ and $\lambda_2 = -0.5076\ldots$ so the figure-eight is linearly stable. These values agree with Simó's up to 8 decimal places.

Question: Can we estimate these values without a computer?
Answer: Maybe. Using a Runge-Kutta-Fehlberg method with local truncation error of order four, only four steps are required to rigorously conclude stability (local truncation error 0.004).

Comparison with Lagrange Solution

The Lagrange solution has the same time-reversal symmetry as the eight. Can do a similar reduction with the same matrix S to obtain a monodromy matrix of the form

$$(SC^{-1}SC)^*$$

Same trick as with the figure-eight to reduce to

$$\frac{1}{2}(\Lambda D + D\Lambda) = \begin{bmatrix} a & 0 \\ 0 & a^{\mathsf{T}} \end{bmatrix}$$

Obtain $a^{\mathsf{T}}(t)$ as a function of time

$$\begin{bmatrix} \cos(2\omega t) & 0 & 0 \\ 0 & f(t) & -g(t) \\ 0 & g(t) & f(t) \end{bmatrix}$$

where $f(t) = \cos(2\omega t)\cosh(\sqrt{2}\omega t), g(t) = \sin(2\omega t)\sinh(\sqrt{2}\omega t)$.
Discriminant of 2×2 block is $-4\sin^2(2\omega t)\sinh^2(\sqrt{2}\omega t)$ which implies instability.

Other Symmetric Orbits: Chen, Ouyang and Xia, (Broucke,Hénon)

Consider absolutely continuous paths γ such that

$$\gamma(t+T) = e^{i\theta}\gamma(t) \qquad \forall t, \text{with } \theta \in (0, 2\pi).$$

Problem: Minimizers in this space are planar relative equilibria.

Solution: Impose a further constraint $\gamma(t) = \overline{\gamma(-t)}$ (time-reversal reflection symmetry).

Configuration must be collinear at times $t = 0$ and $t = T/2$. If θ is a rational multiple of 2π, then minimizers will be periodic solutions.

Chen, Ouyang and Xia (COX) prove that minimizers are attained and collision-free in this special path space. Moreover, there is an open set of n positive masses for which the minimizing solutions are NOT collinear relative equilibria.

Stability of the COX Orbits

The two symmetries on the shape sphere are almost trivial!

$$\gamma(t+T) = \gamma(t) \qquad S = I$$

and

$$\gamma(-t+T) = S\gamma(t) \qquad S = \text{diag}\,\{1, 1, -1, -1, -1, 1\}$$

Theorem

Any periodic orbit in the 3-body problem satisfying the above two symmetries is spectrally stable iff the eigenvalues of $SA^{-1}SA$ are on the unit circle, where $A = X(\frac{T}{2})$.

As with the figure-eight orbit, a nice change of coordinates exists to reduce the problem to a 2×2 matrix.

Conclusions and Questions

- Computing Floquet multipliers by hand is hard, but symmetry and good coordinates make it significantly easier.
- The figure-eight orbit is amazing!
- What is the connection (if any) between being an action-minimizer over a portion of the orbit and the linear stability of the full orbit? (two degrees of freedom model?)
- Is time-reversal symmetry required for stability?
- Can we prove the linear instability of choreographies with n bodies on a figure-eight ($n \geq 5$, n odd)?
- Linear stability of the COX orbits?

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