Saari's Conjecture for the Restricted Three-Body Problem

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The Planar, Circular, Restricted 3-Body Problem (PCR3BP)

$\mathbf{q}_1 = (1-\mu, 0), m_1 = \mu$ and $\mathbf{q}_2 = (-\mu, 0), m_2 = 1-\mu \quad (0 < \mu \leq 1/2)$

Let $a = \sqrt{(x-1+\mu)^2 + y^2}, \quad b = \sqrt{(x+\mu)^2 + y^2}$.

Equations of motion:

$$\begin{aligned} \dot{x} &= u \\ \dot{y} &= v \\ \dot{u} &= V_x + 2v \\ \dot{v} &= V_y - 2u \end{aligned}$$

where

$$V(x,y) = \frac{1}{2}(x^2+y^2) + \frac{\mu}{a} + \frac{1-\mu}{b} + \frac{1}{2}\mu(1-\mu)$$

is the **amended potential.**

Jacobi integral: $E = \frac{1}{2}(u^2+v^2) - V \quad \Longrightarrow \quad V(x,y) \geq -E$

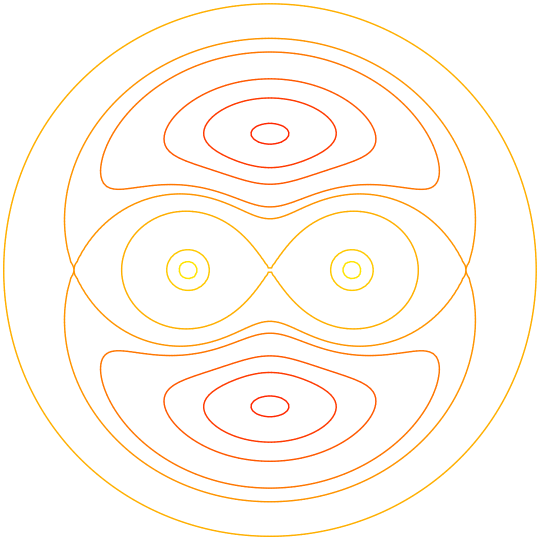

Figure: Level curves of V for $\mu = 1/2$ (equal mass) in the PCR3BP.

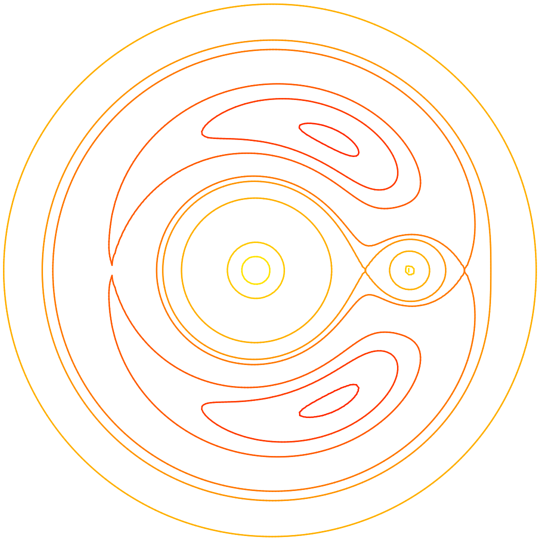

Figure: Level curves of V for $\mu = 0.1$ in the PCR3BP.

Theorem

(GR, LM 2007) The only solutions to the planar, circular, restricted three-body problem (PCR3BP) with a constant value of the amended potential V are equilibria (libration points).

Corollary

(GR, LM 2007) It is not possible for a solution to the PCR3BP to travel with constant speed without being fixed at one of the libration points.

Proof of Corollary: Due to the Jacobi integral, constant speed implies constant potential V. □

Saari's Conjecture (1970) *Every solution of the Newtonian n-body problem that has a constant moment of inertia (constant size) is a relative equilibrium (rigid rotation).*

Fact: Constant inertia $\Rightarrow$ constant potential

$\Rightarrow$ constant kinetic energy

Results on Saari's Conjecture

- Newtonian 3-body problem, equal mass case: Saari's conjecture is true (McCord 2004)
- Newtonian 3-body problem, general case: Saari's conjecture is true (Moeckel 2005)
- Newtonian 3-body problem, any dimension: Saari's conjecture is true (Moeckel 2005)
- Mutual distance potentials, collinear case: Generalized Saari's conjecture is true (Diacu, Pérez-Chavela, Santoprete 2004)
- 5-body problem for certain potentials, and a negative mass: Generalized Saari's conjecture is false (GR 2006)
- Inverse Square potential: Generalized Saari's conjecture is decidedly false

Two polynomial equations in a, b

Suppose $V = c/2$. Then

$$\begin{aligned} V &= c/2 \\ u^2 + v^2 &= k \\ V_x u + V_y v &= 0 \\ \ddot{V} &= 0 \end{aligned}$$

can be reduced to a system of two polynomial equations in the distance variables a and b:

$$\begin{aligned} V &= c/2 \\ ||\nabla V||^8 - 4k||\nabla V||^6 + 2k\Lambda||\nabla V||^4 + k^2\Lambda^2 &= 0 \end{aligned} \qquad (1)$$

where $\Lambda = V_x^2 V_{yy} - 2V_x V_y V_{xy} + V_y^2 V_{xx}$.
Top equation:

$$\mu a^3 b + (1-\mu)ab^3 - cab + 2(1-\mu)a + 2\mu b = 0$$

Bottom equation: 404 terms requiring 30 8.5×11 pages to render
Goal: Show there are only a finite number of solutions to system (1).

BKK Theory

Given $f \in \mathbb{C}[z_1, \ldots z_n], f = \sum c_k z^k, \quad k = (k_1, k_2, \ldots, k_n)$.

The **Newton polytope** of f, denoted $N(f)$, is the convex hull in $\mathbb{R}^n$ of the set of all exponent vectors occurring for f.

Given $\alpha = (\alpha_1, \ldots, \alpha_n)$ with $\alpha_i \in \mathbb{Q}$, the **reduced polynomial** f_α is the sum of all terms of f whose exponent vectors k satisfy

$$\alpha \cdot k = \min_{l \in N(f)} \alpha \cdot l.$$

This equation defines a face of the polytope $N(f)$ with inward pointing normal α.

Let $\mathbb{T} = (\mathbb{C}^*)^n$ where $\mathbb{C}^* = \mathbb{C} - \{0\}$.

Theorem

(Bernstein, 1975) Suppose that system (2) has infinitely many solutions in $\mathbb{T}$. Then there exists a vector $\alpha = (\alpha_1, \ldots, \alpha_n)$ with $\alpha_i \in \mathbb{Q}$ and $\alpha_j = 1$ for some j, such that the system of reduced equations (3) also has a solution in $\mathbb{T}$ (all components nonzero).

$$\begin{aligned} f_1(z_1, \ldots, z_n) &= 0 \\ f_2(z_1, \ldots, z_n) &= 0 \\ &\vdots \\ f_m(z_1, \ldots, z_n) &= 0, \end{aligned} \tag{2}$$

$$\begin{aligned} f_{1\alpha}(z_1, \ldots, z_n) &= 0 \\ f_{2\alpha}(z_1, \ldots, z_n) &= 0 \\ &\vdots \\ f_{m\alpha}(z_1, \ldots, z_n) &= 0. \end{aligned} \tag{3}$$

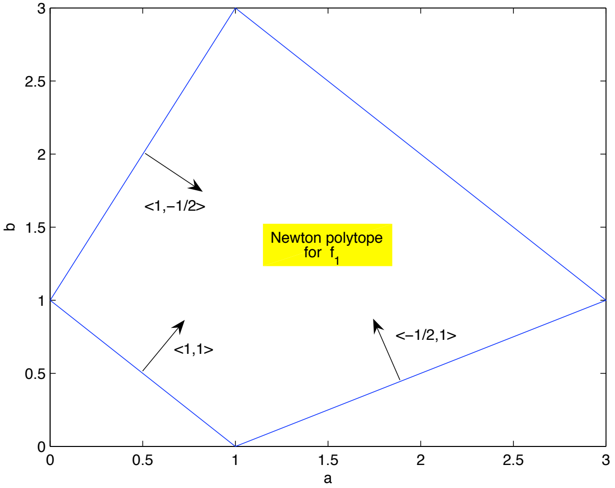

Figure: The Newton polytope corresponding to
$\mu a^3 b + (1-\mu)ab^3 - cab + 2(1-\mu)a + 2\mu b = 0.$

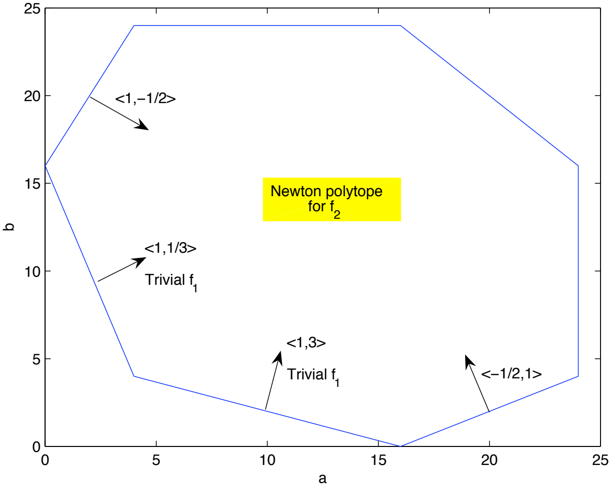

Figure: The Newton polytope corresponding to $||\nabla V||^8 - 4k||\nabla V||^6 + 2k\Lambda||\nabla V||^4 + k^2\Lambda^2 = 0$ (curvature equation).

Only Three Vectors to Consider

1. $\alpha =< 1, -1/2 >$: Inward normal for both polytopes

$$\begin{aligned} b((1-\mu)ab^2 + 2\mu) &= 0 \\ 16\mu^4 b^{16}(-(1-\mu)ab^2 + \mu)^4 &= 0. \end{aligned}$$

   Since $b \neq 0$, substitute $-(1-\mu)ab^2 = 2\mu$ from the first equation into the second to obtain

$$16\mu^4 b^{16}(3\mu)^4 = 0 \qquad \text{only has the trivial solution } b = 0.$$

2. $\alpha =< 1, 1 >$: Gives a point in the second Newton polytope with reduced equation $16(\mu(1-\mu)ab)^4 = 0$.
3. $\alpha =< -1/2, 1 >$: Two reduced equations simplify to

$$1296(1-\mu)^8 a^{16} = 0. \qquad \text{QED}$$

Remarks and Future Work

1. All of the above calculations can be done by hand!
2. Using the volumes (areas) of the polytopes and that of the Minkowski sum gives an exact count of 104 for the number of solutions to our two polynomial equations. Only 4 of these are real, positive solutions, corresponding to the equilibria of the PCR3BP.
3. Bernstein's Theorem does not always succeed. It fails to show the number of equilibria is finite for the PCR3BP since the system $\{V_x = 0, V_y = 0\}$ written as polynomials in a and b has nontrivial solutions along two faces.
4. Next problem: Does the same result hold for the PCR4BP? This is more difficult due to the additional primary and lack of restrictions on the masses. Saari's conjecture for $n = 4$?
5. Additional problem: PCRnBP with equal mass primaries on a regular n-gon. Applications to the charged n-body problem?