Stability of Relative Equilibria in the Planar *n*-Vortex Problem: A Topological Approach

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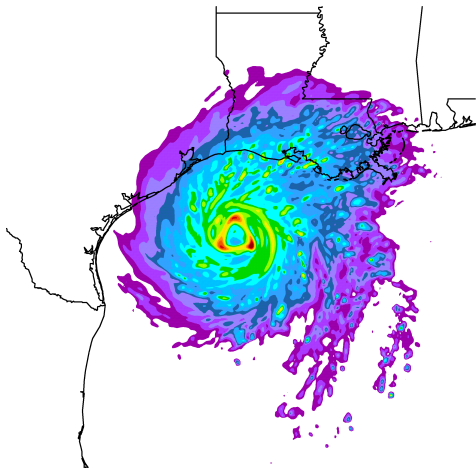Victoria, BC, Canada
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Figure : Weather research and forecasting model from the National Center for Atmospheric Research (NCAR) showing the field of precipitable water for Hurricane Rita (2005). Note the presence of three maxima near the vertices of an equilateral triangle contained within the hurricane's "polygonal" eyewall.
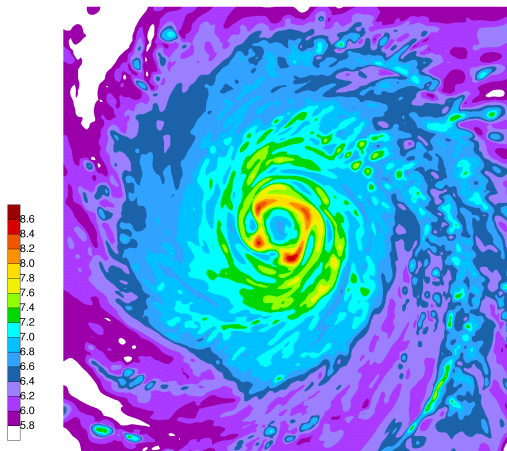http://www.atmos.albany.edu/facstaff/kristen/wrf/wrf.html

Figure : Another NCAR image from the same weather model of Hurricane Rita, this time showing the presence of four "mesavortices."
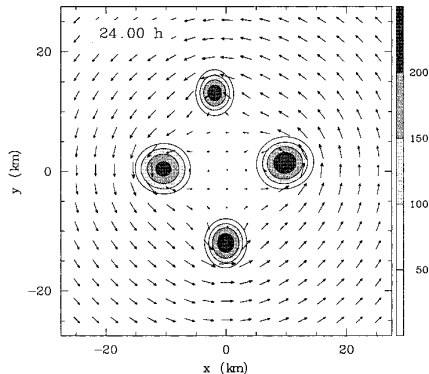
Figure : Result of a numerical simulation carried about by Kossin and Schubert to model the evolution of very thin annular rings of enhanced vorticity in a 2D barotropic framework ("Mesovortices, Polygonal Flow Patterns, and Rapid Pressure Falls in Hurricane-Like Vortices," Kossin and Schubert, *Journal of Atmospheric Sciences*, 2001.) Note the "vortex crystal" of four vortices located close to a rhombus configuration. Darker shading indicates higher vorticity. The configuration shown lasted for the final 18 hours of a 24-hour simulation.
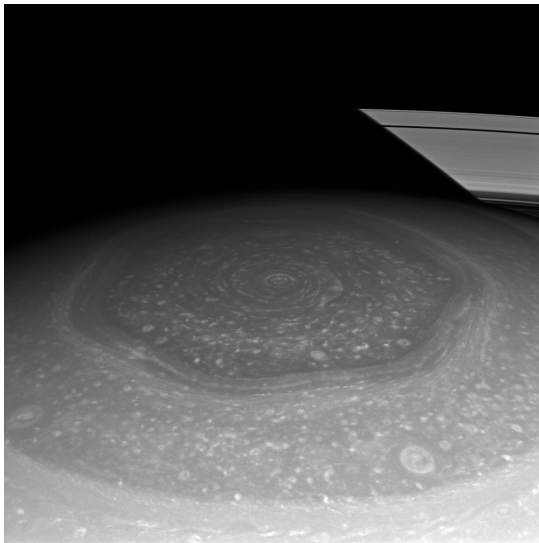
Figure : Saturn's North Pole and its encircling hexagonal cloud structure.
First photographed by Voyager in the 1980's and here again recently by the
Cassini spacecraft – a remarkably stable structure!

Description of the *n*-Vortex Problem

- Introduced by Helmholtz (1858) to model a two-dimensional slice of columnar vortex filaments. Later refined by Lord Kelvin (1867) and Kirchoff (1876).

- Widely used model providing finite-dimensional approximations to vorticity evolution in fluid dynamics.

- General goal is to track the motion of the point vortices rather than focus on their internal structure and deformation, a concept analogous to the use of "point masses" in celestial mechanics.

- Generally "easier" than the *n*-body problem, e.g., the planar three-vortex system is integrable.

- Many techniques used to study the *n*-body problem work perfectly well (often even better!) in the *n*-vortex problem.

The Planar *n*-Vortex Problem: Equations of Motion

A system of *n* planar point vortices with vortex strength $\Gamma_i \neq 0$ and positions $z_i \in \mathbb{R}^2$ evolves according to

$$\Gamma_i \dot{z}_i = J \nabla_i H = J \sum_{j \neq i}^{n} \frac{\Gamma_i \Gamma_j}{r_{ij}^2}(z_j - z_i), \quad 1 \leq i \leq n$$

where

$$H = -\sum_{i<j} \Gamma_i \Gamma_j \ln(r_{ij}), \quad r_{ij} = \|z_i - z_j\|, \quad J = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

and ∇_i denotes the two-dimensional partial gradient with respect to z_i.

Note: Unlike the Newtonian *n*-body problem, $\Gamma_i < 0$ is allowable. The equations do *not* come from $F = ma$.

Special Solutions: Relative Equilibria

Definition

A *relative equilibrium* is a solution of the form

$$z_i(t) = c + e^{-J\omega t}(z_i(0) - c), \quad 1 \leq i \leq n,$$

that is, a uniform rotation with angular velocity $\omega \neq 0$ around some point $c \in \mathbb{R}^2$. Without loss of generality, we take $c = 0$.

The initial positions $z_i(0)$ must satisfy

$$-\omega \, \Gamma_i z_i(0) \;=\; \nabla_i H \;=\; \sum_{j \neq i}^{n} \frac{\Gamma_i \Gamma_j}{r_{ij}^2}(z_j(0) - z_i(0)), \quad 1 \leq i \leq n.$$

We will assume the *total circulation* $\Gamma = \sum_i \Gamma_i \neq 0$. The center of rotation $c = 0$ is equivalent to the *center of vorticity*, $c = \frac{1}{\Gamma} \sum_i \Gamma_i z_i = 0$.

A Topological Approach

The *angular impulse* is the quantity

$$I = \frac{1}{2} \sum_{i=1}^{n} \Gamma_i \|z_i\|^2,$$

the analog of the *moment of inertia* in the *n*-body problem. *I* is a conserved quantity in the planar *n*-vortex problem.

Let $z = (z_1, z_2, \ldots, z_n) \in \mathbb{R}^{2n}$ be the vector of positions. Then the equations defining a relative equilibrium can be written simply as
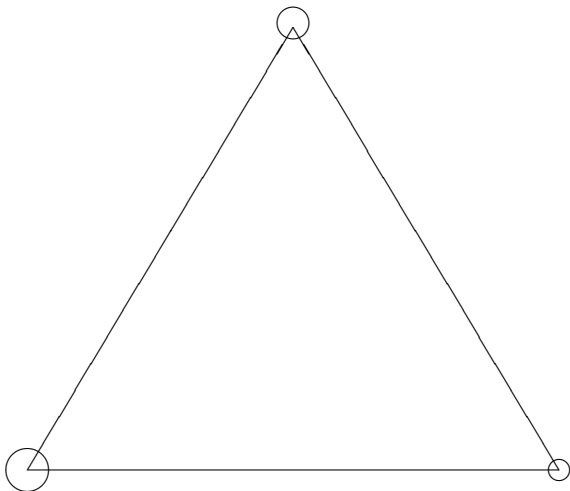
$$\nabla H(z) + \omega \nabla I(z) = 0.$$

Key Fact: A relative equilibrium is a critical point of *H* restricted to a level surface of *I*, with ω serving as the Lagrange multiplier.
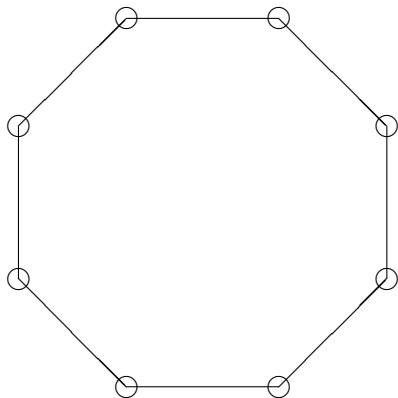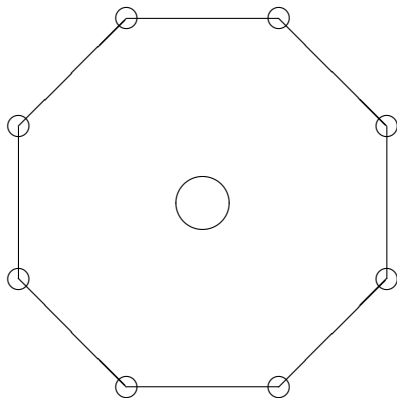
3-Vortex Collinear Configurations (Gröbli 1877)

Equilateral Triangle (Lord Kelvin 1867, Gröbli 1877)

Regular *n*-gon (equal vorticities required for $n \geq 4$)

Some Known Stability Results

- The equilateral triangle solution is linearly (and nonlinearly) stable if and only if $L = \Gamma_1\Gamma_2 + \Gamma_1\Gamma_3 + \Gamma_2\Gamma_3 > 0$ (Lord Kelvin 1867, Gröbli 1877, Synge 1949).

- The regular *n*-gon (equal circulations) is linearly stable only for $n \leq 6$ (Thomson 1883, Havelock 1931, Aref 1995). It is actually degenerate when $n = 7$ although Cabral and Schmidt (1999) showed that it is locally Lyapunov stable by calculating the higher order terms in the normal form of the Hamiltonian.

- By considering a polygonal ring of vortices on the sphere, with additional vortices of arbitrary strength at each pole, Boatto and Simó (2008) reframe the special case $n = 7$ (the Thomson heptagon) as a "bifurcation at infinity."

- For $n \geq 3$, the $1 + n$-gon, with central vortex of circulation κ, is locally Lyapunov stable for κ in a bounded interval whose endpoints depend on *n* (Cabral and Schmidt, 1999).

Some Known Stability Results (cont.)

- Boatto and Cabral (2003) use a sufficient criterion of Dirichlet's (positive or negative definiteness of the Hamiltonian) to deduce linear and nonlinear stability (both at once!) for polygonal rings of identical vortices on the sphere.

- Relative equilibria containing one "dominant" vortex ($\Gamma_0 >> 0$) and n equal strength ($\Gamma_i = \epsilon > 0$) but small vortices are linearly stable provided the n vortices are situated (in the limit $\epsilon \to 0$) at a minimum of a special potential function (Barry, Hall and Wayne, 2012).

- Chen, Kolokolnikov and Zhirov (2013) use an aggregation model and a mean-field limit to find stable relative equilibria in the planar n-vortex problem for large values of n.

1. (Moeckel) Is a linearly stable relative equilibrium of the *n*-body problem always a nondegenerate minimum of the potential function *U* subject to the constraint $I = I_0$?

   This is Problem 16 in "Some problems on the classical *n*-body problem," Albouy, Cabral and Santos, *Celest. Mech. Dyn. Astr* (2012) 113:369–375.

2. Study the linear stability of some families of symmetric relative equilibria in the four-vortex problem that depend on a parameter (e.g., isosceles trapezoid, rhombus).

   Examples calculated in "Relative equilibria in the four-vortex problem with two pairs of equal vorticities," Hampton, GR and Santoprete, to appear in the *Journal of Nonlinear Science*.

Main Results

Theorem

If $\Gamma_i > 0 \; \forall i$, then a relative equilibrium is linearly stable if and only if it is a nondegenerate minimum of H subject to the constraint $I = I_0$.

Two important consequences follow fairly quickly:

- For a generic choice of positive circulations, there exists a non-collinear, linearly stable relative equilibrium.

- For positive circulations, any linearly stable relative equilibrium is also nonlinearly stable (in the sense of Lyapunov).

Our approach follows that of Moeckel's in "Linear Stability Analysis of Some Symmetrical Classes of Relative Equilibria," *Hamiltonian dynamical systems (Cincinnati, OH, 1992)*, IMA Vol. Math. Appl., 63, Springer, 291–317, 1995.

Rotating Coordinates

Let $z = (z_1, z_2, \ldots, z_n) \in \mathbb{R}^{2n}$ be the vector of positions and $M = \operatorname{diag}\{\Gamma_1, \Gamma_1, \ldots, \Gamma_n, \Gamma_n\}$ the $2n \times 2n$ matrix of circulations.

Recall $J = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ and let $K = \operatorname{diag}\{J, J, \ldots, J\}$. The equations of motion are then written compactly as

$$M\dot{z} = K\nabla H(z).$$

Using the matrix $e^{-\omega Jt} = \begin{bmatrix} \cos \omega t & -\sin \omega t \\ \sin \omega t & \cos \omega t \end{bmatrix}$, changing to a uniformly rotating frame with period $2\pi/\omega$ yields the system

$$M\dot{z} = K(\nabla H(z) + \omega Mz).$$

Note: A rest point of the flow, $z_0 : \nabla H(z_0) + \omega Mz_0 = 0$, corresponds to a relative equilibrium solution since $\nabla I(z) = Mz$.

Linear Stability

Linearizing the system in rotating coordinates about a relative equilibrium z_0 gives the stability matrix

$$B = K(M^{-1}D^2H(z_0) + \omega I)$$

where $D^2H(z_0)$ is the Hessian matrix of the Hamiltonian evaluated at the relative equilibrium z_0 and I is the $2n \times 2n$ identity matrix. Here we have used the fact that K and M^{-1} commute.

The eigenvalues of B determine the linear stability of the corresponding periodic solution. Since the system is Hamiltonian, they come in pairs $\pm\lambda$. To have stability, the eigenvalues must lie on the imaginary axis.

Key Properties of *H*

The total vortex angular momentum is $L = \sum_{i<j} \Gamma_i \Gamma_j$.

Recall: $H = -\sum_{i<j} \Gamma_i \Gamma_j \ln(r_{ij})$, $r_{ij} = \|z_i - z_j\|$ and $I = \sum_{i=1}^{n} \Gamma_i \|z_i\|^2$.

1. $\nabla H(z) \cdot z = -L$ (this gives $\omega = L/2I$)

2. $\nabla H(z) \cdot Kz = 0$ (*SO*(2) symmetry – this shows that *I* is conserved)

3. $D^2 H(z) K = -K D^2 H(z)$ (follows from (1) and (2))

Property (3) has profound consequences for the stability analysis. For positive vorticities, it implies a complete factorization of the characteristic polynomial into even quadratic factors.

This property is true for any degree zero homogeneous function *H* with *SO*(2) symmetry.

Consequences of $D^2 H(z) K = -K D^2 H(z)$

Recall: $B = K(M^{-1} D^2 H(z_0) + \omega I)$

- The characteristic polynomials of $D^2 H(z_0)$ and $M^{-1} D^2 H(z_0)$ are even. Moreover, v is an eigenvector with eigenvalue μ if and only if Kv is an eigenvector with eigenvalue $-\mu$.

$$
\begin{aligned}
M^{-1} D^2 H(z_0) v = \mu v &\iff K M^{-1} D^2 H(z_0) v = \mu K v \\
&\iff -M^{-1} D^2 H(z_0) K v = \mu K v \\
&\iff M^{-1} D^2 H(z_0)(Kv) = -\mu(Kv),
\end{aligned}
$$

- If $M^{-1}D^2H(z_0)v = \mu v$, then $Bv = (\mu + \omega)Kv$ and $B(Kv) = (\mu - \omega)v$.

  The matrix B restricted to the invariant subspace $\{v, Kv\}$ takes the simple form

  $$\begin{bmatrix} 0 & \mu - \omega \\ \mu + \omega & 0 \end{bmatrix}$$

  and the characteristic polynomial of B has a quadratic factor of the form $\lambda^2 + \omega^2 - \mu^2$.

- Strategy: Find enough (ideally n) eigenvectors of $M^{-1}D^2H(z_0)$. The resulting eigenvalues μ_i determine the stability of z_0 since the corresponding eigenvalues of B are $\pm\sqrt{\mu_i^2 - \omega^2}$.

Eigenvectors due to Integrals and Symmetry

1. Due to conservation of the center of vorticity, the vectors $s = [1, 0, 1, 0, \ldots, 1, 0]$ and Ks are in the kernel of $D^2 H(z_0)$. Consequently, $\mu = 0$ for the invariant subspace $\{s, Ks\}$ and we obtain the eigenvalues $\pm \omega i$.

2. If z_0 is a relative equilibrium, the vector Kz_0 is in the kernel of B. This is due to the fact that relative equilibria are not isolated (rotational symmetry). The stability matrix B restricted to the invariant subspace $\{z_0, Kz_0\}$ is

$$\begin{bmatrix} 0 & 0 \\ 2\omega & 0 \end{bmatrix}.$$

The off-diagonal term represents the fact that scaling z_0 gives another relative equilibrium but the angular velocity ω must be scaled as well.

Defining Linear Stability

Recall $M = \text{diag}\{\Gamma_1, \Gamma_1, \ldots, \Gamma_n, \Gamma_n\}$. We say that vectors v and w are *M*-orthogonal if $v^T M w = 0$.

Let $V = \text{span}\{z_0, K z_0\}$ and denote $V^\perp \subset \mathbb{R}^{2n}$ as the *M*-orthogonal complement of V. The vector space $V^\perp$ has dimension $2n - 2$ and is invariant under B. Moreover, if $L \neq 0$, $V \cap V^\perp = \emptyset$.

Definition

A relative equilibrium z_0 always has the eigenvalues $0, 0, \pm\omega i$. We call z_0 nondegenerate if the remaining $2n - 4$ eigenvalues are nonzero. It is spectrally stable if the nontrivial eigenvalues are pure imaginary, and linearly stable if, in addition, the restriction of the stability matrix B to $V^\perp$ has a block-diagonal Jordan form with blocks $\begin{bmatrix} 0 & \beta_i \\ -\beta_i & 0 \end{bmatrix}$.

The Special Case $L = 0$

Recall that $\omega = L/2I$ where I is the angular impulse (moment of inertia) and $L = \sum_{i<j} \Gamma_i \Gamma_j$ is the total angular momentum. If a choice of circulations is taken so that $L = 0$, then a critical point z_0 is a(n)

1. equilibrium if $I(z_0) \neq 0$,
2. relative equilibrium if $I(z_0) = 0$.

Theorem (GR, 2013)

Suppose that z_0 is a relative equilibrium for a choice of circulations with $L = 0$. Then two of the nontrivial eigenvalues for z_0 are zero and z_0 is degenerate.

Note that $L = 0$ is only possible if the circulations Γ_i have opposite signs. One would expect interesting bifurcations and a loss of stability to occur when $L = 0$. Two cases where this happens: the equilateral triangle and a rhombus.

Positive Circulations

Suppose that $\Gamma_i > 0 \; \forall i$. Then $\omega = \frac{L}{2I} > 0$ and M is positive definite. We can define an inner product by

$$\langle v, w \rangle \; = \; v^T M w.$$

The matrix $M^{-1} D^2 H(z_0)$ is symmetric with respect to an M-orthonormal basis. Consequently, it has a full set of real eigenvalues of the form $\pm \mu_i$, and the nontrivial eigenvalues of B are all of the form $\pm \sqrt{\mu_i^2 - \omega^2}$.

Theorem (GR, 2013; Kolokolnikov et al., 2013)

If $\Gamma_i > 0 \; \forall i$, then the relative equilibrium z_0 is linearly stable if and only if $|\mu_i| < \omega$ for all nontrivial eigenvalues μ_i of $M^{-1} D^2 H(z_0)$.

Note: If z_0 is linearly stable, then the angular velocity of each component of the linearized flow is $\leq$ the angular velocity of z_0.

Proof of Main Result

Theorem

If $\Gamma_i > 0 \;\forall i$, then a relative equilibrium is linearly stable if and only if it is a nondegenerate minimum of H subject to the constraint $I = I_0$.

Proof (outline): The surface S defined by $I = I_0$ is an ellipsoid. Define the Lagrangian function $G(z) = H(z) + \omega I(z)$. Then, a critical point of G is a relative equilibrium. The type of critical point z_0 is determined by the quadratic form

$$q_{z_0}(v) \;=\; v^T D^2 G(z_0) v \;=\; v^T (D^2 H(z_0) + \omega M)v, \quad v \in T_{z_0} S.$$

We call z_0 a nondegenerate critical point if the nullity of q_{z_0} is one (corresponding to the trivial direction arising from rotation).

If in addition, the index of q_{z_0} is zero, (the dimension of the maximal subspace for which $q_{z_0}(v) < 0$), then z_0 is a nondegenerate minimum of H restricted to S.

Key connection: The stability matrix B is given by

$$B = K(M^{-1}D^2H(z_0) + \omega I)$$

while the quadratic form q_{z_0} is determined by

$$D^2G = D^2H(z_0) + \omega M.$$

Since $\Gamma_j > 0 \ \forall j$, the matrix $M^{-1}D^2H(z_0)$ has an M-orthonormal set of eigenvectors of the form $\{v_j, Kv_j\}$ with eigenvalues $\pm\mu_j$.
For any $v \in T_{z_0}S$ written with respect to this basis, we compute

$$q_{z_0}(v) = \omega(c_2^2 + d_2^2) + \sum_{j=3}^{n}[c_j^2(\mu_j + \omega) + d_j^2(-\mu_j + \omega)].$$

$-\omega < \mu_j < \omega \ \forall j$ implies $q_{z_0}(v) > 0$ for all $v \in T_{z_0}S$ (excluding rotation).

Corollary

For a generic choice of positive circulations, there exists a non-collinear, linearly stable relative equilibrium.

1. The existence of a nondegenerate minimum of H restricted to $I = I_0$ is clear. The fact that the minimum is not collinear is logically expected, but is nontrivial to prove. It follows by generalizing an ingenious idea of Conley's used in the n-body case. Details are in a paper by Pacella (1987).

2. Generically, for any four vortices with positive circulations, there exists a convex, linearly stable, relative equilibrium. This follows by generalizing a clever argument due to Xia (2004).

3. The corollary is false for the n-body problem. If $n \geq 24,306$, then none of the equal-mass relative equilibria are stable (GR, 1999).

Define

$$d(\zeta_0, Z) \ = \ \min\{||\zeta_0 - z|| : z \in Z\}$$

as the distance between a point and a compact set. Let $\gamma = e^{-\omega J t} z_0$ be the periodic solution corresponding to a relative equilibrium z_0.

We call z_0 *nonlinearly stable* or simply *stable* if for any $\epsilon > 0$, there exists a $\delta > 0$ such that $||\zeta_0 - z_0|| < \delta$ implies $d(\phi(t, \zeta_0), \gamma) < \epsilon$ for all $t \in \mathbb{R}$, where $\phi(t, \zeta_0)$ is the solution to the ODE with initial condition ζ_0.

Modifying a proof of Dirichlet's, we have

Theorem (GR, 2013)

Suppose $\Gamma_i > 0 \; \forall i$. Then a relative equilibrium of the planar n-vortex problem is linearly stable if and only if it is nonlinearly stable.

Symmetric Example: Rhombus
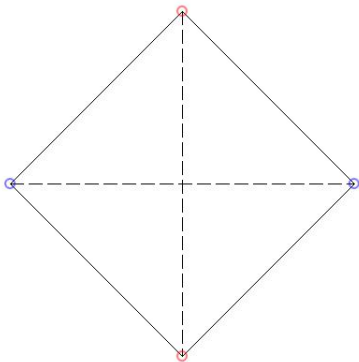
Animation: Two rhombus families of relative equilibria for $\Gamma_1 = \Gamma_2 = 1$ (blue) and $\Gamma_3 = \Gamma_4 = m$ (red). Left family exists for $-1 \leq m \leq 1$; right family exists for $-1 \leq m < 0$. At $m = -2 + \sqrt{3}$, the family on the right is an equilibrium and the direction of rotation flips.

Recall: ω is the angular velocity of the relative equilibrium

Theorem (Hampton, GR, Santoprete, 2013)

There exists two one-parameter families of rhombi relative equilibria with vortex strengths $\Gamma_1 = \Gamma_2 = 1$ and $\Gamma_3 = \Gamma_4 = m$. The vortices 1 and 2 lie on opposite sides of each other, as do vortices 3 and 4. Let $\beta = 3 - 3m$. The mutual distances are given by

$$\left(\frac{r_{34}}{r_{12}}\right)^2 = \frac{1}{2}\left(\beta \pm \sqrt{\beta^2 + 4m}\right), \left(\frac{r_{13}}{r_{12}}\right)^2 = \frac{1}{8}\left(\beta + 2 \pm \sqrt{\beta^2 + 4m}\right),$$
(1)

describing two distinct solutions. Taking $+$ in (1) yields a solution for $m \in (-1, 1]$ that always has $\omega > 0$. Taking $-$ in (1) yields a solution for $m \in (-1, 0)$ that has $\omega > 0$ for $m \in (-2 + \sqrt{3}, 0)$, but $\omega < 0$ for $m \in (-1, -2 + \sqrt{3})$. At $m = -2 + \sqrt{3}$, the $-$ solution becomes an equilibrium. The case $m = 1$ reduces to the square.
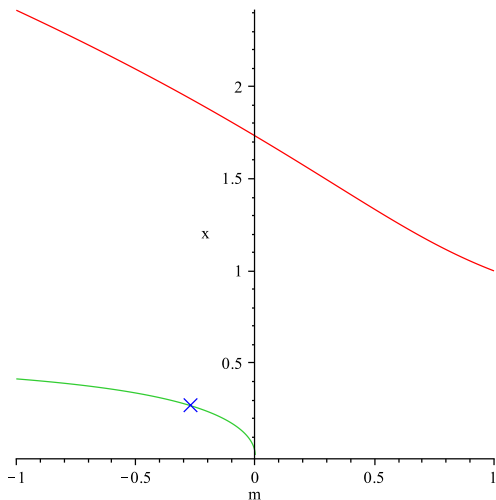
Figure : For the rhombi relative equilibria, take $z_1 = (1, 0)$, $z_2 = -z_1$ and $z_3 = (0, x)$, $z_4 = -z_3$. Then x is a function of m with two branches if $m < 0$. The $\times$ on the bottom curve corresponds to an actual equilibrium; solutions on different sides of this point rotate in opposite directions.

Rhombi: Stability Results

$\Gamma_1 = \Gamma_2 = 1, \Gamma_3 = \Gamma_4 = m$. Set $z_1 = (1,0), z_2 = -z_1$ and $z_3 = (0,y), z_4 = -z_3$. The relation

$$y^2 = \frac{1}{2}\left(\beta \pm \sqrt{\beta^2 + 4m}\right), \quad \beta = 3(1-m)$$

must hold in order for the configuration to be a relative equilibrium.

Using the symmetry of the configuration, it is possible to "guess" the remaining eigenvectors of $M^{-1}D^2H(z_0)$. They are

$$\begin{aligned}
v_1 &= [my, 0, -my, 0, \ 0, -1, 0, 1]^T, \\
v_2 &= [m, 0, m, 0, \ -1, 0, -1, 0]^T
\end{aligned}$$

Note: v_1 corresponds to a perturbation that keeps the symmetry of the configuration but moves the pairs of vortices on the two axes of symmetry in opposite directions (away or towards the origin).

Rhombi: Stability Results (cont.)

Note: The $-$ rhombus family undergoes a pitchfork bifurcation at $m = m^* \approx -0.5951$, where m^* is the only real root of the cubic $9m^3 + 3m^2 + 7m + 5$. As m increases through m^*, the $-$ rhombus solution bifurcates into two convex kite configurations.

Theorem (GR, 2013)

1. *The $+$ rhombus solution is linearly stable for $-(2 - \sqrt{3}) < m \leq 1$. At $m = -(2 - \sqrt{3}) \approx -0.268$, $L = 0$ and the relative equilibrium is degenerate. For $-1 < m < -(2 - \sqrt{3})$, the nontrivial eigenvalues consist of a real pair and a pure imaginary pair.*

2. *The $-$ rhombus solution is always unstable. One pair of eigenvalues is always real. The other pair of eigenvalues is pure imaginary for $-1 < m < m^*$ and real for $m^* < m < 0$. At $m = m^*$, the $-$ rhombus solution is degenerate.*

Symmetric Example: Isosceles Trapezoid
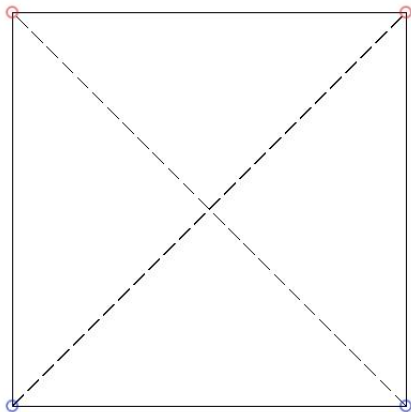
Animation: The isosceles trapezoid family of relative equilibria
($\Gamma_1 = \Gamma_2 = 1$ (blue) and $\Gamma_3 = \Gamma_4 = m$ (red), with $0 < m \leq 1$).

Theorem (Hampton, GR, Santoprete (2013))

There exists a one-parameter family of isosceles trapezoid relative equilibria with vortex strengths $\Gamma_1 = \Gamma_2 = 1$ and $\Gamma_3 = \Gamma_4 = m$. The vortices 1 and 2 lie on one base of the trapezoid, while 3 and 4 lie on the other. Let $\alpha = m(m+2)/(2m+1)$. If $r_{13} = r_{24}$ are the lengths of the two congruent diagonals, then the mutual distances are described by

$$\left(\frac{r_{34}}{r_{12}}\right)^2 = \alpha, \quad \left(\frac{r_{14}}{r_{12}}\right)^2 = \frac{1}{2}\left(m+2-\sqrt{\alpha}\right)$$

$$\text{and} \quad \left(\frac{r_{13}}{r_{12}}\right)^2 = \frac{1}{2}\left(m+2+\sqrt{\alpha}\right).$$

This family exists if and only if $m > 0$. The case $m = 1$ reduces to the square. For $m \neq 1$, the larger pair of vortices lie on the longest base.

Isosceles Trapezoid: Stability Results

$\Gamma_1 = \Gamma_2 = 1, \Gamma_3 = \Gamma_4 = m$. Set $z_1 = (1, 0), z_2 = -z_1$ and
$z_3 = (-x, y), z_4 = (x, y)$. To be a relative equilibrium, we require

$$(x, y) = (\sqrt{\alpha}, \sqrt{2m + 3 - \alpha}) \quad \text{where} \quad \alpha = \frac{m(m + 2)}{2m + 1}.$$

Surprisingly, the vector $u1 = [mx, 0, -mx, 0, 1, 0, -1, 0]^T$ is in the kernel of $D^2 H(z_0)$. This means that two of the nontrivial eigenvalues are $\pm \omega i$.

The other eigenvector of $M^{-1} D^2 H(z_0)$ is of the form
$u_2 = [-a, -m, a, -m, ax, 1, -ax, 1]^T$ where $a = \frac{m\sqrt{3(2m+1)}}{m^2 + m + 1}$.

Theorem (GR, 2013)

The isosceles trapezoid family is stable for all values of the parameter $m > 0$. The nontrivial eigenvalues are $\pm \omega i$ and $\frac{\pm \sqrt{3L}}{2(m + 2)} i$.

Comments/Future Work

- For $\Gamma_1 = \Gamma_2 = 1, \Gamma_3 = \Gamma_4 = m$ and $-2 + \sqrt{3} < m < 0$, the $+$ rhombus solution is linearly stable. However, it is actually a saddle of the Hamiltonian restricted to $I = I_0 > 0$. Thus, for opposite signed circulations it is possible to have a linearly stable relative equilibrium and not be a minimum.

- For $\Gamma_1 = \Gamma_2 = 1, \Gamma_3 = \Gamma_4 = m$ and $m > 0$, there is a unique convex central configuration for a given ordering of vortices. Can we prove uniqueness for any choice of four positive circulations?

  Note: Uniqueness does not hold if $m < 0$ (e.g., two rhombi solutions).

- Implications for fluid dynamics? Can we say anything about stability (meta-stability) for a PDE model??

- Stability of other families in the four-vortex problem (e.g., asymmetric configurations, kites, collinear with $m < 0$)?
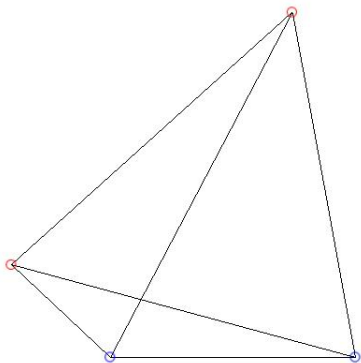
Animation: The asymmetric family of relative equilibria ($\Gamma_1 = \Gamma_2 = 1$ (blue) and $\Gamma_3 = \Gamma_4 = m$ (red), with $-1 < m \leq 1$). The configuration is concave for $m > 0$ and convex for $m < 0$.
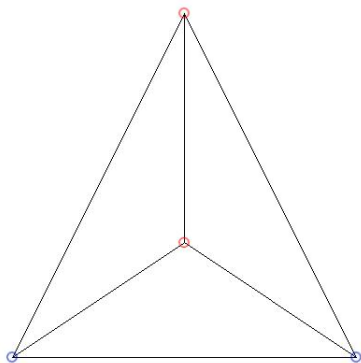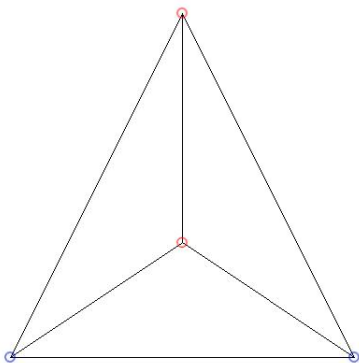
Animation: Two kite families of relative equilibria ($\Gamma_1 = \Gamma_2 = 1$ (blue) and $\Gamma_3 = \Gamma_4 = m$ (red)). The family on the left exists for $0 < m \leq 1$, while the one on the right exists for $-1/2 < m \leq 1$. The configuration on the right is concave for $m > 0$ and convex for $m < 0$.