Relative Equilibria in the Four-Vortex Problem with Two Pairs of Equal Vorticities

Gareth E. Roberts

Department of Mathematics and Computer Science
College of the Holy Cross

Marshall Hampton (University of Minnesota, Duluth)
Manuele Santoprete (Wilfrid Laurier University)

NSF DMS-1211675

New Trends in Dynamical Systems
Salou (Tarragona), Spain
October 1–5, 2012

The Planar n-Vortex Problem: Equations of Motion

A system of n planar point vortices with vortex strength $\Gamma_i \neq 0$ and positions $x_i \in \mathbb{R}^2$ evolves according to

$$\Gamma_i \dot{x}_i \;=\; J \nabla_i H \;=\; -J \sum_{j \neq i}^{n} \frac{\Gamma_i \Gamma_j}{r_{ij}^2}(x_i - x_j), \quad 1 \leq i \leq n$$

where

$$H = -\sum_{i<j} \Gamma_i \Gamma_j \ln(r_{ij}), \quad r_{ij} = \|x_i - x_j\|, \quad J = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

and ∇_i denotes the two-dimensional partial gradient with respect to x_i.

Note: Unlike the Newtonian n-body problem, $\Gamma_i < 0$ is allowable. The equations do *not* come from $F = ma$.

Description of the n-Vortex Problem

- Introduced by Helmholtz (1858) to model a two-dimensional slice of columnar vortex filaments. Later refined by Lord Kelvin (1867) and Kirchoff (1876).

- Widely used model providing finite-dimensional approximations to vorticity evolution in fluid dynamics.

- General goal is to track the motion of the point vortices rather than focus on their internal structure and deformation, a concept analogous to the use of "point masses" in celestial mechanics.

- Generally "easier" than the n-body problem, e.g., the planar three-vortex system is integrable.

- Many techniques used to study the n-body problem work perfectly well (sometimes even better) in the n-vortex problem.

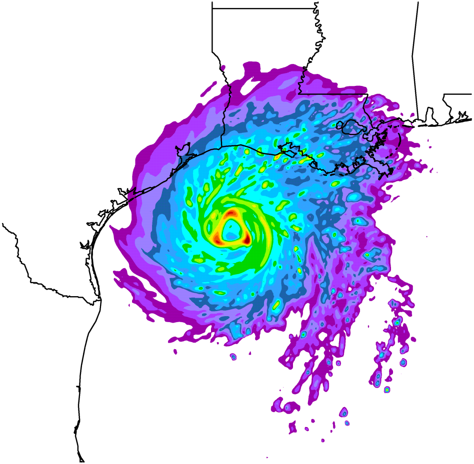


Figure: Weather research and forecasting model from the National Center for Atmospheric Research (NCAR) showing the field of precipitable water for Hurricane Rita (2005). Note the presence of three maxima near the vertices of an equilateral triangle contained within the hurricane's "polygonal" eyewall.
`http://www.atmos.albany.edu/facstaff/kristen/wrf/wrf.html`

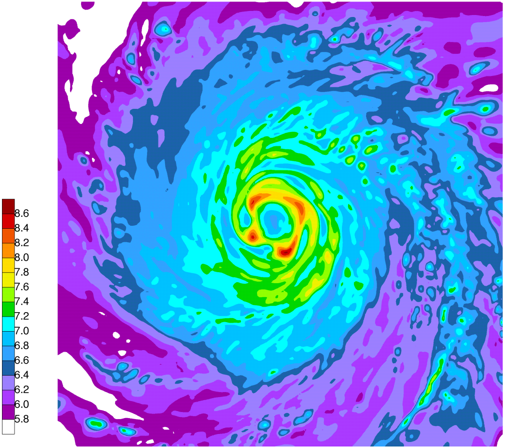


Figure: Another NCAR image from the same weather model of Hurricane Rita, this time showing the presence of four “mesavortices.”

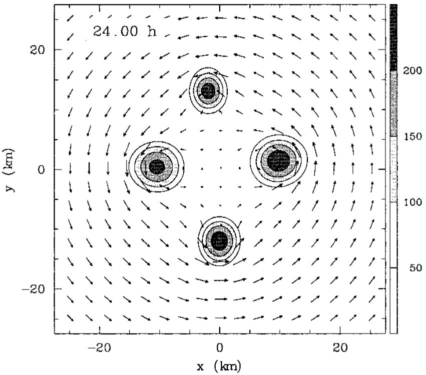


Figure: Result of a numerical simulation carried about by Kossin and Shubart to model the evolution of very thin annular rings of enhanced vorticity in a 2D barotropic framework ("Mesovortices, Polygonal Flow Patterns, and Rapid Pressure Falls in Hurricane-Like Vortices," Kossin and Shubert, *Journal of Atmospheric Sciences*, 2001.) Note the "vortex crystal" of four vortices located close to a rhombus configuration. Darker shading indicates higher vorticity. The flow pattern shown lasted for about 18 hours.

Special Solutions: Relative Equilibria

Definition

A *relative equilibrium* is a solution of the form

$$x_i(t) = c + e^{-J\lambda t}(x_i(0) - c), \quad 1 \leq i \leq n,$$

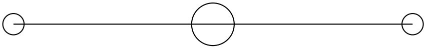
that is, a uniform rotation with angular velocity $\lambda \neq 0$ around some point $c \in \mathbb{R}^2$.

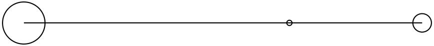
The initial positions $x_i(0)$ must satisfy

$$-\lambda(x_i(0) - c) \; = \; \frac{1}{\Gamma_i}\nabla_i H \; = \; \sum_{j \neq i}^{n} \frac{\Gamma_j}{r_{ij}^2}(x_j(0) - x_i(0)), \quad 1 \leq i \leq n.$$

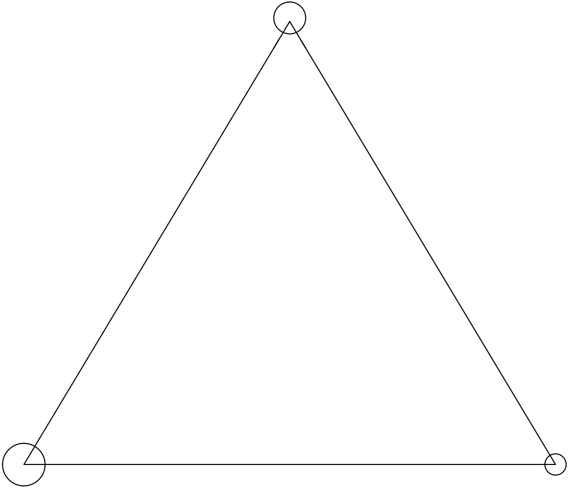
If the *total circulation* $\Gamma = \sum_i \Gamma_i \neq 0$, then the center of rotation c must be the *center of vorticity*, $c = \frac{1}{\Gamma}\sum_i \Gamma_i x_i$.

3-Vortex Collinear Configurations (Gröbli 1877)

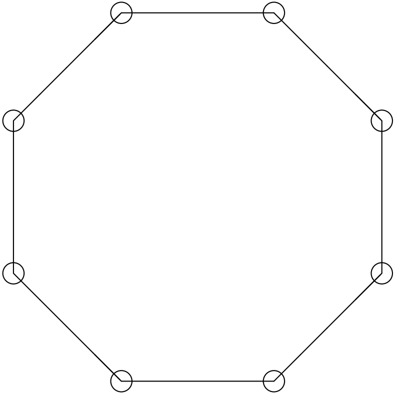




Equilateral Triangle (Lord Kelvin 1867, Gröbli 1877)



Regular n-gon (equal vorticities required for $n \geq 4$)



Four-Vortex Relative Equilibria with Two Pairs of Equal Vorticities

Goal: Classify all 4-vortex relative equilibria with circulations $\Gamma_1 = \Gamma_2 = 1$ and $\Gamma_3 = \Gamma_4 = m$, where $-1 < m \leq 1$ is a parameter.

- While there are some numerical studies for $n \geq 4$, few analytic results exist. Specifying the equality in the vortex strengths helps make the problem tractable.
- Due to the work of Hampton and Moeckel (2009), the number of strictly planar (planar but not collinear) relative equilibria is at most 74 (up to symmetry) and the number of collinear relative equilibria is at most 12. Finiteness of relative equilibria equivalence classes and the upper bounds are obtained using BKK theory.
- This problem was partially motivated by the companion problem in celestial mechanics where it is unproven that a 4-body convex relative equilibrium with two pairs of adjacent equal masses must possess a line of symmetry.

Shape and Symmetry of Configurations

A configuration of four vortices that is not collinear (nor contain any three vortices which are collinear) can be classified as either concave or convex.

1. Concave: one vortex located strictly inside the convex hull of the other three (i.e., a triangle with a point in the interior).

2. Convex: no vortices contained inside the convex hull of the other three (i.e., a convex quadrilateral).

Symmetry: Given the symmetry in the choice of vorticities, $\Gamma_1 = \Gamma_2 = 1$ and $\Gamma_3 = \Gamma_4 = m$, are solutions always symmetric? How does the symmetry and the shape of the solution vary with m?

A configuration is called a kite if two vortices are on an axis of symmetry and the other two vortices are symmetrically located with respect to this axis. Kite configurations may either be concave or convex.

Major Results

1. Enumeration: Precise counts are obtained for the number of solutions (equivalence classes) in terms of the shape and type of configuration as a function of m.

2. Symmetry Theorem: Any convex relative equilibrium with $m > 0$, and any concave solution with $m < 0$, must have a line of symmetry. For the convex case, the symmetric solutions are a rhombus and an isosceles trapezoid. In the concave case, the symmetric solution is an isosceles triangle with an interior vortex on the axis of symmetry (a concave kite).

3. Bifurcations: Interesting bifurcations are found at $m = 1, 0, -1/2$ in terms of the number and type of solutions. At $m = m^* \approx -0.5951$, the only real root of $9m^3 + 3m^2 + 7m + 5$, a family of rhombi undergoes a pitchfork bifurcation, giving birth to a special family of convex kite solutions.

Mathematical Techniques Employed

1. Computational Algebraic Geometry: e.g., computing Gröbner bases of ideals, using elimination theory or the extension theorem. Symbolic computations were performed using Sage, Singular and Maple. No numerical approximations required to prove results.
2. Analysis: e.g., examining the roots of a polynomial with coefficients in m as m varies

$$\begin{aligned}\zeta(z) = \; & m^2(m+2)(1+2m)^2 z^4 \\ & -4m(15m^4+61m^3+91m^2+61m+15)\, z^3 \\ & +(300m^5+1508m^4+2910m^3+2696m^2+1188m+200)\, z^2 \\ & -4(5m+4)(25m^4+127m^3+231m^2+175m+45)\, z \\ & +(m+2)^3.\end{aligned}$$

$m = 2/5$

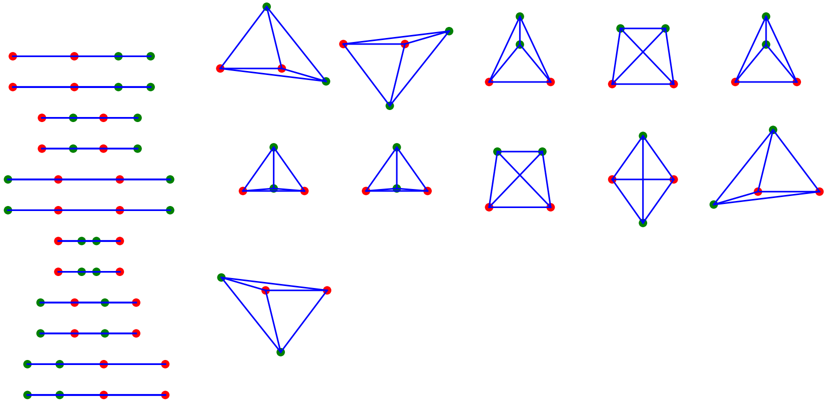


Figure: The full set of solutions for $m = 2/5$. Vortices $\Gamma_1 = \Gamma_2 = 1$ are denoted by red disks and vortices $\Gamma_3 = \Gamma_4 = m$ by green ones.

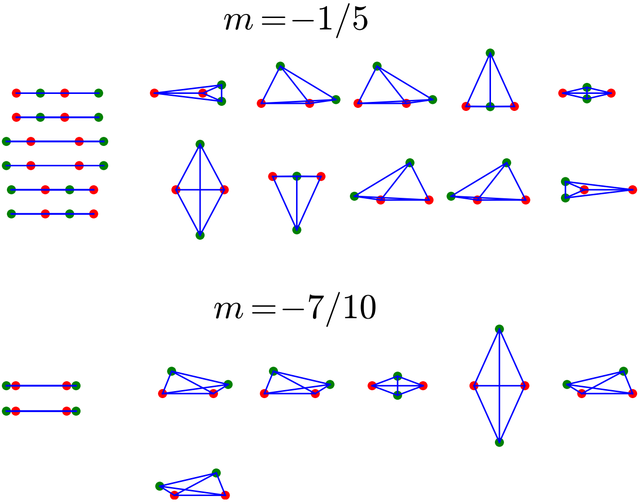


Figure: The full set of solutions for $m = -1/5$ and $m = -7/10$. Vortices $\Gamma_1 = \Gamma_2 = 1$ are denoted by red disks and vortices $\Gamma_3 = \Gamma_4 = m$ by green ones.

Shape	$m \in (-1, 1]$	Type of solution (number of)
Convex	$m = 1$	Square (6)
	$0 < m < 1$	Rhombus (2), Isosceles Trapezoid (4)
	$-1 < m < 0$	Rhombus (4) Asymmetric (8)
	$-1/2 < m < 0$	Kite_{34} (4)
	$m^* < m < -1/2$	Kite_{12} (4)
Concave	$m = 1$	Equi. Triangle with Interior Vortex (8)
	$0 < m < 1$	Kite_{34} (8) Asymmetric (8)
	$-1/2 < m < 0$	Kite_{12} (4)
Collinear	$m = 1$	Symmetric (12)
	$0 < m < 1$	Symmetric (4) Asymmetric (8)
	$-1 < m < 0$	Symmetric (2)
	$-1/2 < m < 0$	Asymmetric (4)

$m \in (-1, 1]$	Number of solutions (equiv. clases)
$m = 1$	26
$0 < m < 1$	34
$-1/2 < m < 0$	26
$m = -1/2$	14
$m^* < m < -1/2$	18
$-1 < m \leq m^*$	14

Table: The number of relative equilibria equivalence classes as a function of m. Recall that $m^* \approx -0.5951$ is the only real root of $9m^3 + 3m^2 + 7m + 5$.

Note: The presence of the singular bifurcation at $m = -1/2$ is likely a consequence of the fact that the sum of three vorticities vanishes here, a particularly troubling case when attempting to prove finiteness.

The Bifurcation at $m = 1$

- Three distinct configurations, all symmetric: square (6), equilateral triangle with a vertex at the center (8), collinear solution (12). This is different than the Newtonian 4-body problem where Albouy showed there are four geometrically distinct relative equilibria.

- The equilateral triangle with interior vortex is a highly degenerate solution of the four-vortex problem. The Hessian matrix of the defining equations has a null space of dimension 2 (excluding the eigenvector in the direction of rotation).

- If vortex 3 or 4 is at the center of the triangle, then the equilateral triangle solution bifurcates into two different isosceles triangles with the interior vortex on the line of symmetry (concave kites).
- If vortex 1 or 2 is at the center of the triangle, the solution branches into two asymmetric concave configurations that are identical under a reflection.

Mutual Distances Make Great Coordinates

Use the six mutual distances $r_{12}, r_{13}, r_{14}, r_{23}, r_{24}, r_{34}$ as variables.

The initial positions of a relative equilibrium can be found as critical points of

$$H - \lambda(I - I_0) - \frac{\mu}{32} e_{CM}$$

where I is the moment of inertia with respect to the center of vorticity, $I = \frac{1}{2\Gamma} \sum_{i<j} \Gamma_i \Gamma_j r_{ij}^2$, and e_{CM} is the Cayley-Menger determinant

$$e_{CM} = \begin{vmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & r_{12}^2 & r_{13}^2 & r_{14}^2 \\ 1 & r_{12}^2 & 0 & r_{23}^2 & r_{24}^2 \\ 1 & r_{13}^2 & r_{23}^2 & 0 & r_{34}^2 \\ 1 & r_{14}^2 & r_{24}^2 & r_{34}^2 & 0 \end{vmatrix} = 0.$$

Equations for a Four-Vortex Relative Equilibrium

Differentiating with respect to each of the six mutual distance variables gives

$$\begin{aligned}
\Gamma_1\Gamma_2(r_{12}^{-2} + \lambda') &= \sigma A_1 A_2, & \Gamma_3\Gamma_4(r_{34}^{-2} + \lambda') &= \sigma A_3 A_4 \\
\Gamma_1\Gamma_3(r_{13}^{-2} + \lambda') &= \sigma A_1 A_3, & \Gamma_2\Gamma_4(r_{24}^{-2} + \lambda') &= \sigma A_2 A_4 \\
\Gamma_1\Gamma_4(r_{14}^{-2} + \lambda') &= \sigma A_1 A_4, & \Gamma_2\Gamma_3(r_{23}^{-2} + \lambda') &= \sigma A_2 A_3
\end{aligned}$$

where $\lambda' = \lambda/\Gamma$, $\sigma = 2\mu$ and A_i is the oriented area of the triangle whose vertices contain all the vortices except for the i-th vortex.

This yields the well-known Dziobek (1900) equations (but for vortices)

$$(r_{12}^{-2} + \lambda')(r_{34}^{-2} + \lambda') = (r_{13}^{-2} + \lambda')(r_{24}^{-2} + \lambda') = (r_{14}^{-2} + \lambda')(r_{23}^{-2} + \lambda').$$

Symmetry Theorem: Outline of Proof

1. Show that any solution satisfies: $r_{13} = r_{24}$ if and only if $r_{14} = r_{23}$, and $r_{ij} = r_{ik}$ if and only if $r_{lj} = r_{lk}$ where i, j, k, l are distinct indices.
2. Compute a Gröbner basis for the ideal generated by the Albouy-Chenciner (1997) equations (both symmetric and un-symmetric), the Dziobek equations and the Cayley-Menger determinant, all of which are polynomials in the variables $s_{ij} = r_{ij}^2$.
3. Saturate this basis with respect to $(s_{13} - s_{24}), (s_{14} - s_{23}), \ldots$ to eliminate solutions with a line of symmetry.
4. Use a term order that eliminates all variables except s_{12} and s_{13} to find $s_{12} = (2m+1)/(m+1)$ or $s_{12} = 1/(m+1)$, and a fourth-degree polynomial $p(s_{13})$ with coefficients in m and s_{12}.
5. Substitute the values of s_{12} into the coefficients of p and analyze the roots of the resulting polynomials in terms of m. (Hard part!)
6. Show that if $m > 0$, the only possible solutions are concave, and that if $m < 0$, the only possible solutions are convex.

Symmetric Example: Isosceles Trapezoid

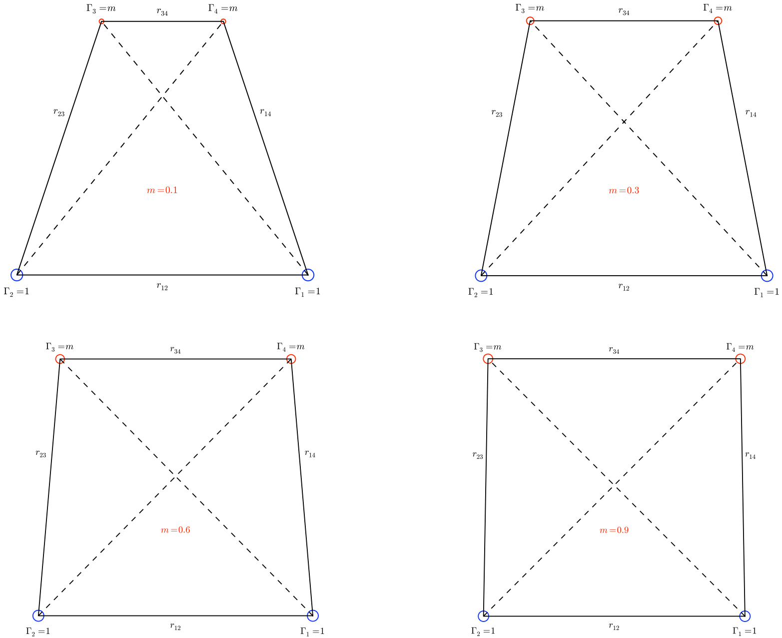
Theorem

There exists a one-parameter family of isosceles trapezoid relative equilibria with vortex strengths $\Gamma_1 = \Gamma_2 = 1$ *and* $\Gamma_3 = \Gamma_4 = m$. *The vortices 1 and 2 lie on one base of the trapezoid, while 3 and 4 lie on the other. Let* $\alpha = m(m+2)/(2m+1)$. *If* $r_{13} = r_{24}$ *are the lengths of the two congruent diagonals, then the mutual distances are described by*

$$\left(\frac{r_{34}}{r_{12}}\right)^2 = \alpha, \quad \left(\frac{r_{14}}{r_{12}}\right)^2 = \frac{1}{2}\left(m + 2 - \sqrt{\alpha}\right)$$

$$\text{and} \quad \left(\frac{r_{13}}{r_{12}}\right)^2 = \frac{1}{2}\left(m + 2 + \sqrt{\alpha}\right).$$

This family exists if and only if $m > 0$. *The case* $m = 1$ *reduces to the square. For* $m \neq 1$, *the larger pair of vortices lie on the longest base.*



$\Gamma_3 = m$
r_{34}
$\Gamma_4 = m$
r_{23}
r_{14}
$m = 0.1$
$\Gamma_2 = 1$
r_{12}
$\Gamma_1 = 1$
$\Gamma_3 = m$
r_{34}
$\Gamma_4 = m$
r_{23}
r_{14}
$m = 0.3$
$\Gamma_2 = 1$
r_{12}
$\Gamma_1 = 1$
$\Gamma_3 = m$
r_{34}
$\Gamma_4 = m$
r_{23}
r_{14}
$m = 0.6$
$\Gamma_2 = 1$
r_{12}
$\Gamma_1 = 1$
$\Gamma_3 = m$
r_{34}
$\Gamma_4 = m$
r_{23}
r_{14}
$m = 0.9$
$\Gamma_2 = 1$
r_{12}
$\Gamma_1 = 1$

Symmetric Example: Rhombus

Recall: λ is the angular velocity of the relative equilibrium

Theorem

There exists two one-parameter families of rhombi relative equilibria with vortex strengths $\Gamma_1 = \Gamma_2 = 1$ *and* $\Gamma_3 = \Gamma_4 = m$*. The vortices 1 and 2 lie on opposite sides of each other, as do vortices 3 and 4. Let* $\beta = 3 - 3m$*. The mutual distances are given by*

$$\left(\frac{r_{34}}{r_{12}}\right)^2 = \frac{1}{2}\left(\beta \pm \sqrt{\beta^2 + 4m}\right), \left(\frac{r_{13}}{r_{12}}\right)^2 = \frac{1}{8}\left(\beta + 2 \pm \sqrt{\beta^2 + 4m}\right), \tag{1}$$

describing two distinct solutions. Taking $+$ *in (1) yields a solution for* $m \in (-1, 1]$ *that always has* $\lambda > 0$*. Taking* $-$ *in (1) yields a solution for* $m \in (-1, 0)$ *that has* $\lambda > 0$ *for* $m \in (-2 + \sqrt{3}, 0)$*, but* $\lambda < 0$ *for* $m \in (-1, -2 + \sqrt{3})$*. At* $m = -2 + \sqrt{3}$*, the* $-$ *solution becomes an equilibrium. The case* $m = 1$ *reduces to the square.*

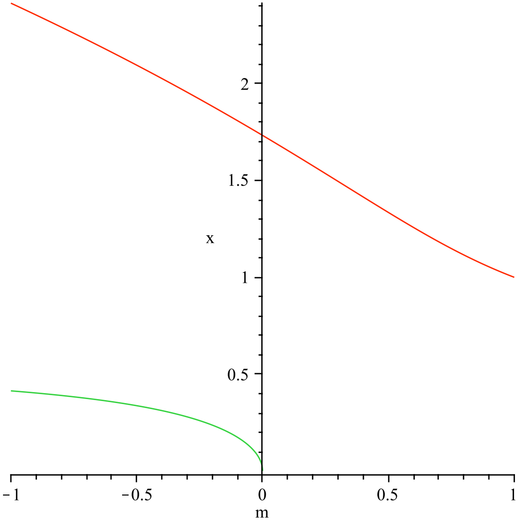


Figure: For the rhombi relative equilibria, $x = r_{34}/r_{12}$ is a function of m with two branches if $m < 0$.

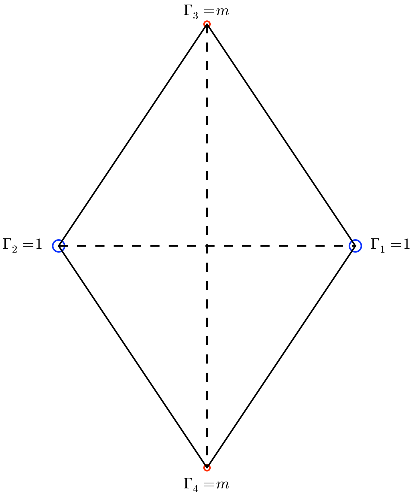
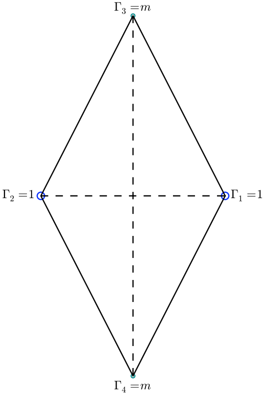


Figure: The rhombus relative equilibrium with $m = 0.3$.



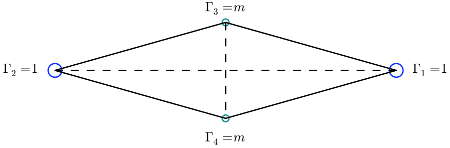


Figure: The two distinct rhombi relative equilibria when $m = -0.3$. The solutions rotate in opposite directions.

A Pitchfork Bifurcation

- Let $m^* \approx -0.5951$ denote the only real root of the cubic $9m^3 + 3m^2 + 7m + 5$.
- As m increases through m^*, the $-$ rhombus solution bifurcates into two convex kite solutions with the positive strength vortices on the axis of symmetry. The two kites are distinguished by whether $r_{13} > r_{23}$ or $r_{13} < r_{23}$.
- Since the rhombus solution continues to exist past the bifurcation, we have a pitchfork bifurcation.
- The Hessian matrix $D^2(H + \lambda I)$ evaluated at the $-$ rhombus solution at $m = m^*$ has a null space of dimension 1 (excluding the "trivial" eigenvector in the direction of rotation) and contains an eigenvector corresponding to a perturbation in the direction of the convex kite solution.

Future Work

- Generalize to the 4-body problem. Is there a similar type of symmetry theorem that is provable? Can we perturb away from the equal mass square and show that no bifurcations occur in the convex case for $0 < m \leq 1$?

- Linear stability analysis of the solutions found. In particular, focus on the symmetric families. How does the stability change passing through a bifurcation? What about non-linear stability?

- Assuming that stable solutions exist, find applications. Are these solutions of any real, physical interest?

- Thank you for your attention!