Undergraduate Research in Conceptual Climate Modeling

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Background

- Cara Donovan (HC ’18): undergraduate math major with a minor in computer science.
- Research started in summer of 2017 (HC Summer Research Program) and continued through the year as a College Honors thesis.
- Cara had little training in ODE’s, dynamical systems, or mathematical modeling, but she knew how to program.
- Both of us interested in learning about climate science and low-dimensional mathematical models of the Earth’s climate.
- Interdisciplinary project (physics, geology, chemistry, statistics): Cara’s thesis readers were a geologist and a statistician.

Figure: Cara Donovan and myself after her senior thesis presentation at Holy Cross (May 2, 2018).

Budyko's Energy Balance Model (1969)

Basic idea: Temperature is driven by differences between energy coming in (solar radiation) and going out (outgoing longwave radiation)

Variables: $y = \sin\theta$, where θ is the usual latitude, $y \in [0, 1]$
$T = T(t, y)$, the mean annual temperature at "latitude" y.

$$R\,\frac{\partial T}{\partial t} \;=\; Q\,s(y)(1-\alpha) - (A + BT) - C(T - \overline{T})$$

$$\begin{aligned}
Q &= \text{solar constant} \approx 342\ \text{W/m}^2 \\
s(y) &= \text{insolation distribution (quadratic)} \\
\alpha &= \text{albedo (reflectivity of planet)} \\
A + BT &= \text{outgoing longwave radiation} \\
C(T - \overline{T}) &= \text{meridional heat transport} \\
\overline{T} &= \int_0^1 T(t, y)\,dy \;=\; \text{mean global annual temp.}
\end{aligned}$$

Albedo and the Ice Line

Albedo varies with latitude, depending on whether the surface is snow-covered ice, land, sea ice, or water.

Define the parameter $\eta \in [0, 1]$ to be the ice line, the latitudinal boundary between snow-covered ice and water.

Two-step albedo function:

$$\alpha(y; \eta) = \begin{cases} \alpha_w & \text{if } y < \eta \\ \alpha_s & \text{if } y > \eta, \end{cases}$$

where $\alpha_w \approx 0.32$ (water) and $\alpha_s \approx 0.62$ (snow-covered ice).

We will assume that $T_c = -10°\text{C}$ is the critical temperature at which glaciers can form.

Equilibrium solutions

Let $T^* = T^*(y;\eta)$ represent the equilibrium solution of the Budyko model. Integrating the right-hand side of the ODE with respect to y from 0 to 1 yields the global mean temperature at equilibrium

$$\overline{T^*} = \frac{1}{B}(Q(1-\overline{\alpha}(\eta)) - A),$$

where $\overline{\alpha}(\eta) = \int_0^1 s(y)\alpha(y;\eta)\,dy$ is the weighted average albedo.

This in turn gives a formula for the equilibrium temperature profile

$$T^* = T^*(y;\eta) = \frac{Q}{B+C}\left(s(y)(1-\alpha(y,\eta)) + \frac{C}{B}(1-\overline{\alpha}(\eta))\right) - \frac{A}{B}.$$

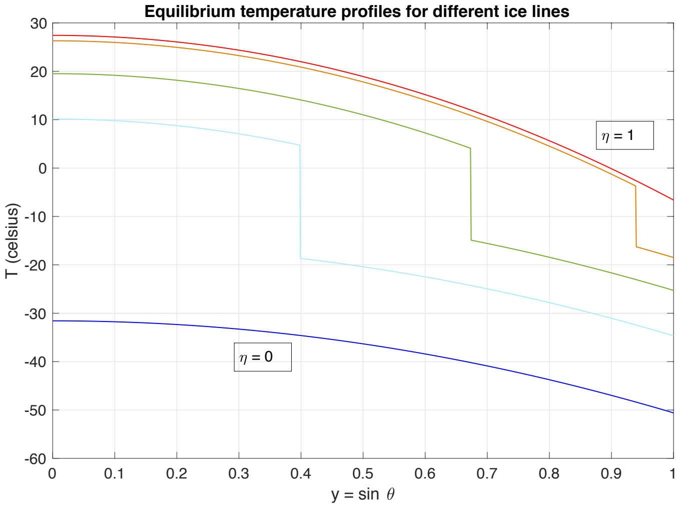


Figure: Graphs of equilibrium temperature profiles with two-step albedo function for different ice lines: $\eta = 1$ (red; ice free), $\eta = \sin(70^\circ)$ (orange; current), $\eta = \sin(42.3^\circ)$ (green; Worcester), $\eta = \sin(23.5^\circ)$ (light blue; Tropic of Cancer), $\eta = 0$ (blue; snowball).

Evolution of a Snowball Earth Event ...

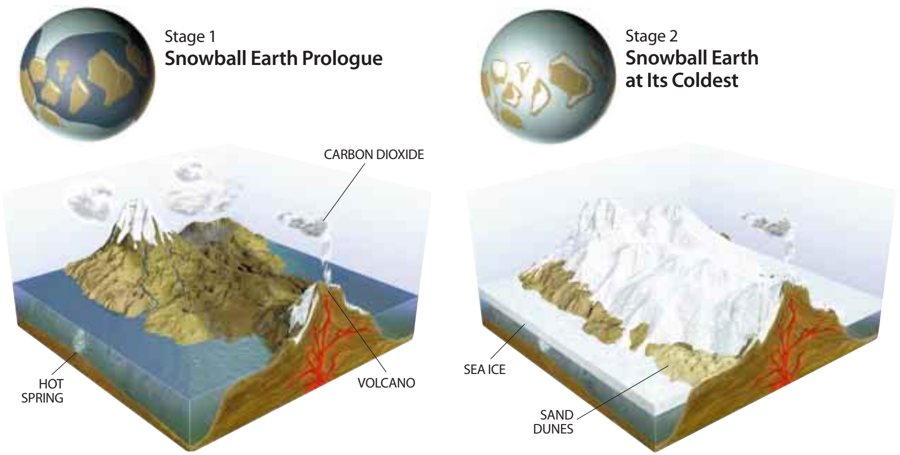
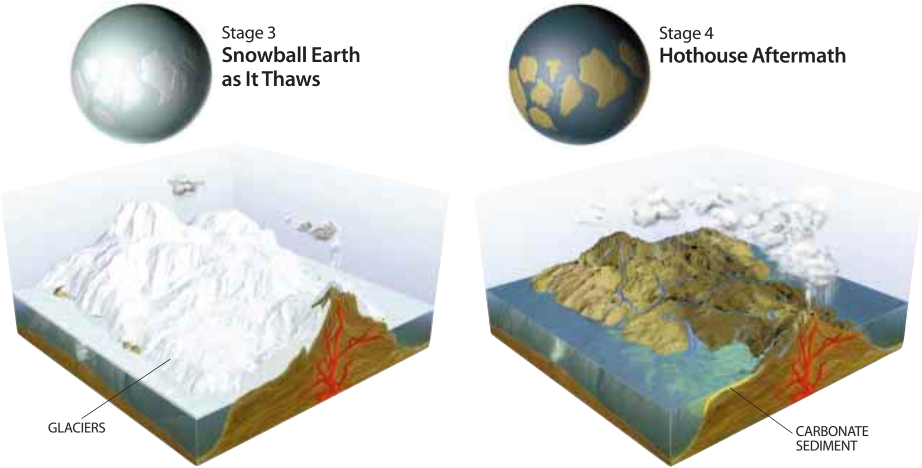


Figure: Ample geological evidence suggests the Earth was almost entirely covered by glaciers twice in the Neoproterozoic era (about 700 mya). (Hoffman and Schrag, "Snowball Earth," *Scientific American*, Jan. 2000)

… AND ITS HOTHOUSE AFTERMATH



Widiasih's Extension of Budyko Model (2013)

Recall that $T_c = -10°$C is the critical temperature at which glaciers can form.

Treat the ice line η as a variable and append the ODE

$$\frac{d\eta}{dt} = \epsilon(h(\eta) - T_c)$$

to the Budyko model, where ϵ is a small parameter and

$$\begin{aligned} h(\eta) &= T^*(\eta,\eta) = \frac{1}{2}\left(\lim_{y\to\eta^-} T^*(y,\eta) + \lim_{y\to\eta^+} T^*(y,\eta)\right) \\ &= \frac{Q}{B+C}\left[s(\eta)\left(1 - \frac{\alpha_w + \alpha_s}{2}\right) + \frac{C}{B}(1 - \overline{\alpha}(\eta))\right] - \frac{A}{B} \end{aligned}$$

is the equilibrium temperature at the ice line. This extension models the movement of the ice line and enables a stability analysis of any equilibria.

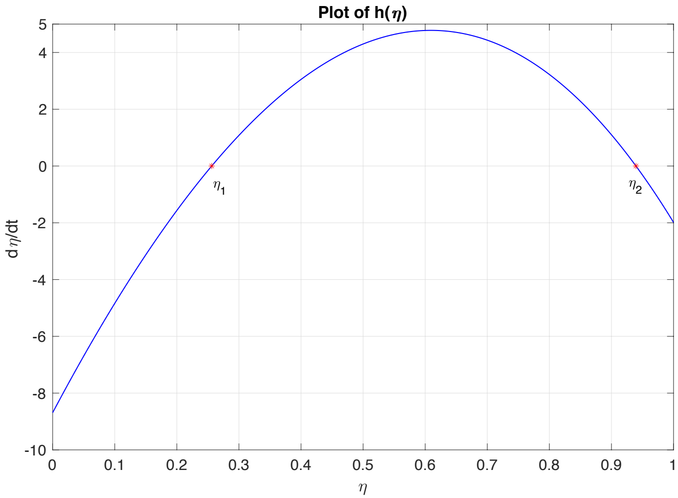


Figure: Plot of $h(\eta) - T_c$ for the Widiasih ice line equation $d\eta/dt = \epsilon(h(\eta) - T_c)$ showing two equilibria ice line positions at $\eta_1 \approx 0.2562$ (unstable) and $\eta_2 \approx 0.9394$ (stable).

Modeling Climate in the Neoproterozoic Era

Assume land is clustered near the equator and introduce a new parameter $\alpha_l \approx 0.4$ to model the albedo of land.

Three-step albedo function: ($y_L \approx 0.35$)

$$\alpha(y,\eta) = \begin{cases} \alpha_l & \text{if } 0 \le y < y_L, \\ \alpha_w & \text{if } y_L < y < \eta, \\ \alpha_s & \text{if } \eta < y \le 1\ . \end{cases}$$

Bifurcation Analysis: What happens as we vary α_s? What if we vary A (a proxy for the amount of CO_2 in the atmosphere) as well?

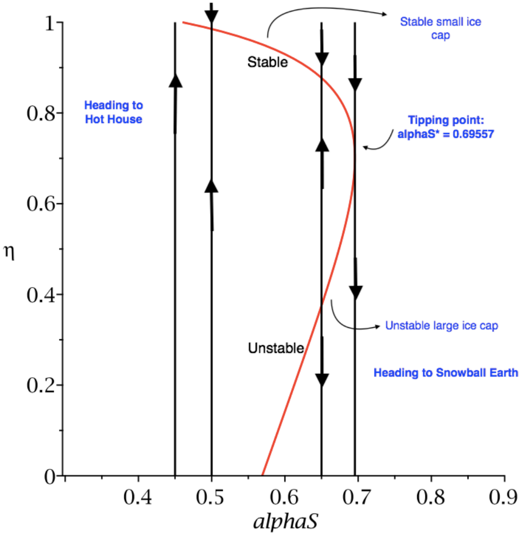


Figure: Bifurcation diagram showing the location of the ice line equilibria (roots of $h(\eta) - T_c$) as the albedo parameter α_s is varied. Note the saddle node bifurcation (tipping point) at $\alpha_s \approx 0.69557$. Figure by Cara Donovan.

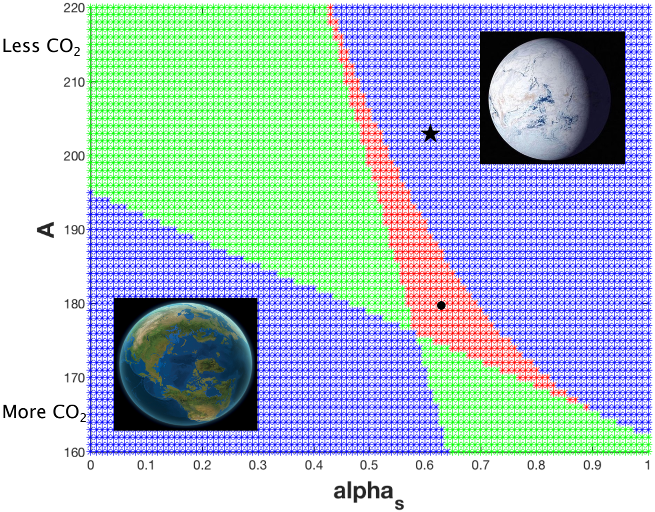


Figure: Two-dimensional bifurcation diagram indicating the number of ice line equilibria as A and α_S are varied. Red means two equilibria (one stable, one unstable); green means one equilibrium (the other root is less than 0 or greater than 1); blue indicates no equilibria. Figure by Cara Donovan.

A Discrete Dynamical Systems Approach

Following the work of Walsh and Widiasih (2014), we approximate the Budyko-Widiasih PDE model by

$$\begin{aligned} T_{n+1}(y) &= T_n(y) + F(T_n(y), \eta_n) \\ \eta_{n+1} &= \eta_n + G(T_n(y), \eta_n), \end{aligned}$$

where

$$\begin{aligned} F(T, \eta) &= \frac{K}{R}\left(Qs(y)(1-\alpha(y,\eta)) - (A+BT) - C(T-\overline{T})\right) \\ G(T, \eta) &= \epsilon(T(\eta) - T_c). \end{aligned}$$

$n \in \mathbb{N} \cup \{0\}$ is in years and $y \in [0, 1]$ represents latitude, as before.

Given an initial temperature profile $T_0(y)$ and an initial ice line η_0, iterations of the system give next year's temperature and ice line.

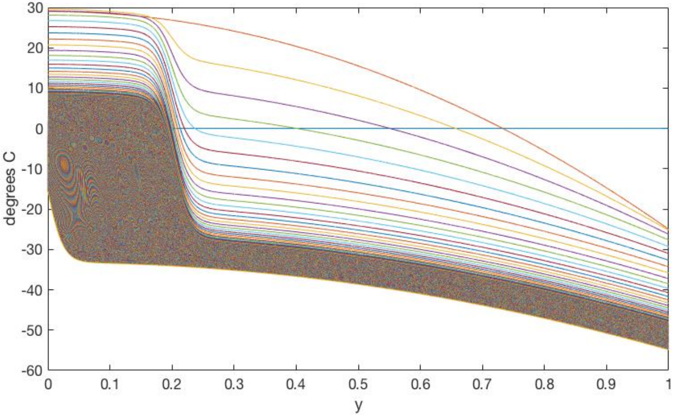


Figure: Iterations of the coupled Budyko-Widiasih model under Neoproterozoic conditions with $\epsilon = 10^{-5}$, $\eta_0 = 0.2$, $\alpha_s = 0.65$, and $A(\eta) = \frac{220}{7}\eta + \frac{1216}{7}$. Simulation took 7653 iterations to reach Snowball state ($\eta = 0$). Figure created by Cara Donovan using Matlab.

Adjustments to the Model: Conditional Albedo

Assuming a band of land around the equator for $y \in [0, y_L]$, we define two albedo functions depending on the location of the ice line η.

1. If $\eta > y_L$ (planet is land, water, and snow-covered ice), then

$$\alpha_1(y,\eta) = \begin{cases} \alpha_l & \text{if } 0 \le y < y_L \\ \frac{\alpha_s+\alpha_w}{2} + \frac{\alpha_s-\alpha_w}{2}\tanh(M(y-\eta)) & \text{if } y_L \le y \le 1. \end{cases}$$

2. If $0 \le \eta \le y_L$ (planet is land and snow-covered ice), then

$$\alpha_2(y,\eta) = \frac{\alpha_s+\alpha_l}{2} + \frac{\alpha_s-\alpha_l}{2}\tanh(M(y-\eta)).$$

We also vary A with η to reflect a decline in silicate weathering:

$$A(\eta) = \begin{cases} 200 & y_L < \eta < 1 \\ 133.\overline{3}\,\eta + 153.\overline{3} & 0 < \eta \le y_L. \end{cases}$$

Some Results: Heading to Snowball

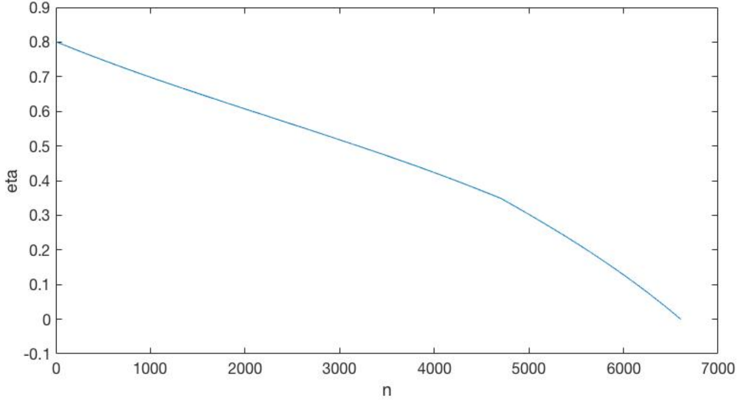


Figure: Simulation of the ice line η under Neoproterozoic conditions with $\epsilon = 10^{-5}$ and $\eta_0 = 0.8$. Snowball Earth is reached after 6605 iterations. Figure created by Cara Donovan using Matlab.

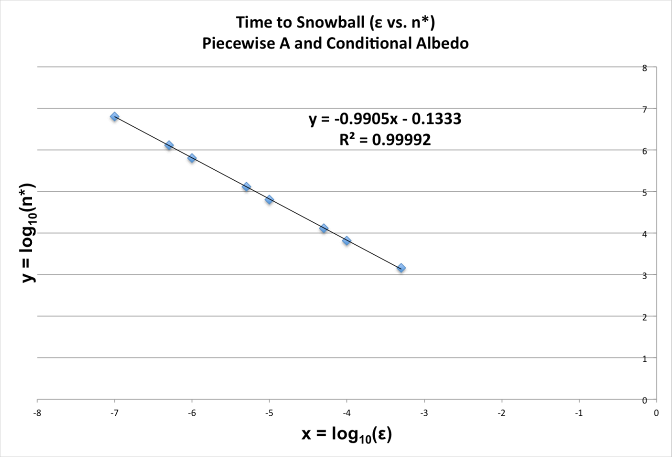


Figure: Log plot of time to reach Snowball state (n^*) versus ϵ under Neoproterozoic conditions. A simple inverse relationship is suggested: $n^* \approx 1/\epsilon$. Figure created by Cara Donovan using Microsoft Excel.

Concluding Remarks

- Low-dimensional energy balance models capture overall climate states quite well. Interesting bifurcations (tipping points) occur.
- Our models and simulations support the theory that land clustered near the equator is a necessary condition for the climate to head toward a Snowball state.
- Plenty of interesting research projects in conceptual climate modeling that are accessible to motivated undergraduates (e.g., climate of other planets, effects of deforestation, impact of climate change)
- Research with undergraduates is rewarding, important, and fun!
- Thank you for your attention!

References

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5. Walsh, J., Widiasih, E., A Dynamics Approach To A Low-Order Climate Model, *Discrete and Continuous Dynamical Systems Series B* **19**, no. 1 (2014), 257–279.