Elusive Zeros Under Newton’s Method

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Newton’s Method

Iterative root-finding method $f(x) = 0$: $x_0, x_1, x_2, \ldots$

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

**Figure:** Newton’s Method for finding a root of a function on $\mathbb{R}$. Image source: http://aleph0.clarku.edu/~djoyce/newton/method.html
Newton’s Method as a Dynamical System

\[ N_p(z) = z - \frac{p(z)}{p'(z)}, \quad p : \mathbb{C} \mapsto \mathbb{C} \]

If \( \alpha \) is a simple root of \( p \), then \( \alpha \) is a super-attracting fixed point for \( N_p \), i.e., \( N_p(\alpha) = \alpha, N'_p(\alpha) = 0 \).

Newton’s method “tends” to obey the nearest-root principal: initial seeds iterate towards the closest root.

If \( p(z) \) is a quadratic polynomial with distinct roots, \( N_p \) is topologically conjugate to \( z \mapsto z^2 \). The Julia set of \( N_p \) is precisely the perpendicular bisector of the line segment connecting the two roots.
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Success of Newton’s Method

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- **Bad**: Points in the Julia set of $N_p$ **never** converge to a root. These are bad places to guess, although a small perturbation of such a guess will still find a root.

- **Ugly**: In certain cases, Newton’s method $N_p$ may contain an extraneous attracting cycle distinct from the roots of $p$. This would yield an entire open **region** of the complex plane that never converges to a root. Here, a small perturbation may not improve your situation!
Figure: The dynamical plane for Newton’s method applied to $p_\lambda(z) = (z - 1)(z + 1)(z - \lambda)(z - \bar{\lambda})$ with $\lambda \approx 0.4438656912 \, i$. The “bad” initial seeds (black) iterate towards a super-attracting period 2-cycle.
The Ugly/Interesting Case

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Theorem (Fatou, Julia): Every attracting cycle of a rational map attracts at least one critical point.

\[ N'(z) = p(z) \cdot p''(z) / [p'(z)]^2, \]

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**Simple Technique:** Follow the orbit of the critical points which are different from the roots. These “free” critical points will lead to an extraneous attracting cycle should it exist. (Curry, Garnett & Sullivan 1983)
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Since $N'_p(z) = \frac{p(z) p''(z)}{[p'(z)]^2}$, the inflection points of $p$ are the free critical points of $N_p$. 
The Cubic Case

\[ p_\lambda(z) = (z - 1)(z + 1)(z - \lambda), \quad \lambda \in \mathbb{C} \]

**Figure:** The parameter plane for Newton’s method applied to \( p_\lambda \). Black parameter values correspond to polynomials for which the free critical point does not converge to a root, i.e., it is drawn into an extraneous attracting cycle.
Research on Cubic Newton Maps

- J. Head (1988)
- S. Sutherland (1989)
- Tan Lei (1990, 1997)
- F. Haesler and H. Kriete (1993)
- P. Blanchard (1994)
- P. Roesch (1997)
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- Theory of polynomial-like mappings
  A. Douady and J. Hubbard (1985)
A Symmetric Fourth-Degree Polynomial Family

\[ p_\lambda(z) = (z - 1)(z + 1)(z - \lambda)(z - \bar{\lambda}), \quad \lambda \in \mathbb{C} \]

\[ = z^4 - 2\text{Re}(\lambda)z^3 + (|\lambda|^2 - 1)z^2 + 2\text{Re}(\lambda)z - |\lambda|^2 \]

Symmetric location of the roots (kite configuration) leads to nice reductions and interesting dynamics.
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Two free critical points: \( p''_\lambda = 0 \)

\[ c_\pm = \frac{1}{2} \left( \text{Re}(\lambda) \pm \sqrt{(\text{Re}(\lambda))^2 - \frac{2}{3}(|\lambda|^2 - 1)} \right) \]

Goal: Follow the orbits of \( c_\pm \) as \( \lambda \) varies. If an extraneous attracting cycle exists, it must attract at least one of these orbits.
If $\lambda = a + bi$, then the discriminant of the quadratic defining the two critical points $c_{\pm}$ is given by

$$\delta = \frac{1}{3} \left( a^2 - 2b^2 + 2 \right).$$
Symmetry

Let $N_{\lambda} = N_{p_{\lambda}}$
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For $\lambda = \beta i$, $N_{\beta i} \sim N_{i/\beta}$. For this interesting case, we can restrict to a complicated 1-d real map with $0 < \beta \leq 1$ (analytic work)
The Case $\lambda = \beta i$

$$N_\beta(x) = \frac{3x^4 + (\beta^2 - 1)x^2 + \beta^2}{4x^3 + 2(\beta^2 - 1)x}.$$  

Free critical points are real and symmetric with respect to the origin. Thus, any extraneous attracting cycle for Newton’s method must lie on the real axis.
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- $N_\beta$ is an odd function.
- For $1/\sqrt{3} \leq \beta < 1$, $c_+$ converges to $-1$ while $c_-$ converges to $1$ under iteration of $N_\beta$. 
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- For odd periods, the free critical points can never lie on the same periodic orbit.
Figure: The orbit diagram for $N_\beta$ with $\beta = (2\sqrt{5} - 3)/\sqrt{11} \approx 0.4438656912$ showing a super-attracting 2-cycle between $c_+$ and $c_-$. 
Figure: The dynamical plane for Newton’s method applied to
\( p_\lambda(z) = (z - 1)(z + 1)(z - \lambda)(z - \bar{\lambda}) \) with \( \lambda \approx 0.4438656912 i \). The “bad”
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<table>
<thead>
<tr>
<th>Per</th>
<th>$\beta$</th>
<th>Type</th>
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<th>Type</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>0.4438657165</td>
<td>Bitransitive</td>
<td>5</td>
<td>0.2296915054</td>
<td>Disjoint</td>
</tr>
<tr>
<td>2</td>
<td>0.3835689425</td>
<td>Disjoint</td>
<td>5</td>
<td>0.2275660932</td>
<td>Disjoint</td>
</tr>
<tr>
<td>3</td>
<td>0.2291103601</td>
<td>Disjoint</td>
<td>5</td>
<td>0.2249682546</td>
<td>Disjoint</td>
</tr>
<tr>
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<td>0.1341462433</td>
<td>Disjoint</td>
<td>5</td>
<td>0.1846443415</td>
<td>Disjoint</td>
</tr>
<tr>
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<td>5</td>
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<td>Disjoint</td>
</tr>
<tr>
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<td>5</td>
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<td>Disjoint</td>
</tr>
<tr>
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<td>Bitransitive</td>
<td>5</td>
<td>0.1289675832</td>
<td>Disjoint</td>
</tr>
<tr>
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<td>Disjoint</td>
<td>5</td>
<td>0.1125293225</td>
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</tr>
<tr>
<td>4</td>
<td>0.1134351641</td>
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<tr>
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<td>5</td>
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<td>Disjoint</td>
</tr>
<tr>
<td>5</td>
<td>0.2299712598</td>
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<td>5</td>
<td>0.0298646167</td>
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**Table:** The table of $\beta$ values for which $N_\beta$ has super-attracting periodic cycles. Also listed is the type of cycle: Bitransitive (free critical points on same orbit) or Disjoint (free critical points on separate orbits).
Figure: The bifurcation diagram for $N_\beta$ showing the asymptotic behavior of both free critical points as a function of $\beta$. The horizontal line segments at the top and bottom of the figure are 1 and $-1$. 
Figure: The $\lambda$-parameter plane for $N_\lambda$ following the orbit of both free critical points (shading indicates different rates of convergence.) The window is $[-1, 1] \times [-i, i]$. 
A Connection to Cubic Maps


Suppose that both critical points are attracted to periodic cycles (not necessarily the same):

- **Bitransitive:** Critical points attracted to same periodic orbit. Obtain swallow configurations and tricorns in a real cross-section of the parameter plane. Prototype models:
  - **Swallow:** \( x \mapsto x^2 + c_1, x_0 \mapsto x_0^2 + c_2, c_1, c_2 \in \mathbb{R} \)
  - **Tricorn:** \( z \mapsto z^2 + c, c \in \mathbb{C} \)

- **Disjoint Periodic Sinks:** Critical points attracted to different periodic orbits. Obtain product configurations and Mandelbrot sets in a real cross-section of the parameter plane. Prototype models:
  - **Product:** \( x \mapsto x^2 + c_1, y \mapsto y^2 + c_2, c_1, c_2 \in \mathbb{R} \)
  - **Mandelbrot Set:** \( z \mapsto z^2 + c, c \in \mathbb{C} \)
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Figure: An example of Milnor’s “swallow configuration” in the parameter plane for $N_\lambda$ centered at the bitransitive value $\lambda \approx 0.443865i$. 
Figure: As expected (according to Milnor), a tricorn is located in the parameter plane at the inversion \((1/\beta) \, i\) of the bitransitive value of the previous figure. In this case, the two free critical points are complex conjugates. The prototype for this case is the map \(z \mapsto (z^2 + c)^2 + \bar{c}\).
Figure: Zooming in on the parameter plane near the a disjoint periodic value, $\lambda \approx 0.2291i$, exhibiting a “product” configuration.
Figure: The Mandelbrot-like set in the parameter plane arising from the inversion \((1/\beta) i\) of our disjoint periodic value of the previous figure.
Some Final Observations

- **Conjecture:** Each bitransitive $\lambda$-value corresponding to the two free critical points sharing the same super-attracting $n$-cycle lies at the center of a swallow configuration in the parameter plane.

The yellow diamond shaped boundary in the parameter plane is defined by those $\lambda$-values where both $p'_\lambda$ and $p''_\lambda$ simultaneously vanish. If $\lambda = a + bi$, this occurs on the algebraic curve $(a^2 - 2b^2 + 2)^3 - 27a^2(b^2 + 1)^2 = 0$.

Taking successive pre-images of this curve appears to define the sequence of intertwining yellow “leaves” that approach the real axis.
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Thank You for Your Attention