On Central Configurations

Gareth E. Roberts

Department of Mathematics and Computer Science
College of the Holy Cross

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Definition

A *central configuration* (c.c.) is a configuration of bodies $(\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n), \mathbf{x}_i \in \mathbb{R}^d$ such that the acceleration vector for each body is a common scalar multiple of its position vector. Specifically, in the Newtonian n-body problem with the center of mass at the origin, for each index i,

$$\sum_{j \neq i}^{n} \frac{m_i m_j (\mathbf{x}_j - \mathbf{x}_i)}{||\mathbf{x}_j - \mathbf{x}_i||^3} + \lambda\, m_i \mathbf{x}_i = 0$$

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- The collinear c.c.'s correspond to $d = 1$, planar c.c.'s to $d = 2$, spatial c.c.'s to $d = 3$. One can also study theoretically the case $d > 3$.
- Summing together the n equations above quickly yields $\sum_i m_i \mathbf{x}_i = 0$.

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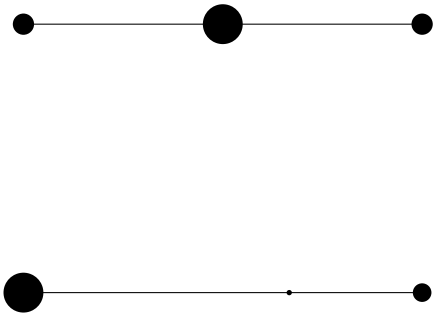
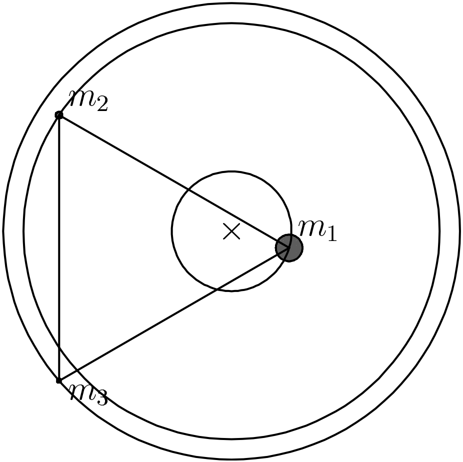
- Bifurcations in the topology of the integral manifolds (holding hc^2 constant where h is the value of the energy and c is the length of the angular momentum vector) occur precisely at values corresponding to central configurations.
- 193 articles found on MathSciNet using a general search for "central configurations"

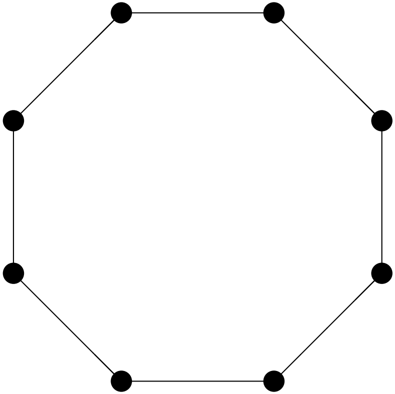
3-Body Collinear Configuration (Euler 1767)



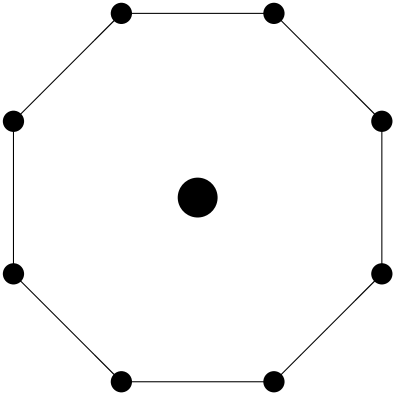
Equilateral Triangle (Lagrange 1772)



Regular n-gon (equal mass required for $n \geq 4$)



$1 + n$-gon (arbitrary central mass)



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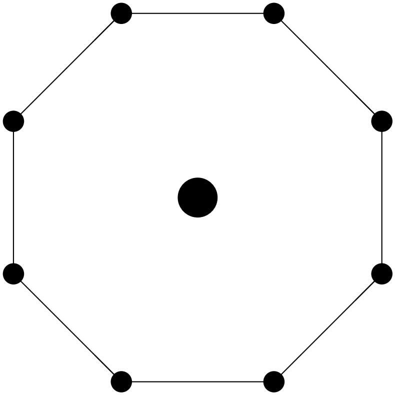

Used by Sir James Clerk Maxwell in 1859 in Stability of the Motion of Saturn's Rings (winner of the Adams Prize)

An Alternate Characterization of CC's

Let $r_{ij} = ||\mathbf{q}_i - \mathbf{q}_j||$ where $\mathbf{q}_i$ denotes the position of the i-th body. The Newtonian potential function is

$$U(\mathbf{q}) = \sum_{i<j}^{n} \frac{m_i m_j}{r_{ij}}$$

The equations of motion for the n-body problem are then given by

$$\begin{aligned} m_i \ddot{\mathbf{q}}_i &= \frac{\partial U}{\partial \mathbf{q}_i}, \qquad i \in \{1, 2, \ldots n\} \\ &= \sum_{j \neq i}^{n} \frac{m_i m_j (\mathbf{q}_j - \mathbf{q}_i)}{r_{ij}^3} \end{aligned}$$

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Consequently, the i-th equation for a c.c. can be written as

$$\frac{\partial U}{\partial \mathbf{q}_i}(\mathbf{x}) + \lambda m_i \mathbf{x}_i = 0.$$

CC's as critical points of U

The moment of inertia $I(\mathbf{q})$ (w.r.t. the center of mass) is defined as

$$I(\mathbf{q}) = \frac{1}{2}\sum_{i=1}^{n} m_i ||\mathbf{q}_i||^2.$$

Thus, the equations for a c.c. can be viewed as a Lagrange multiplier problem (set $I(\mathbf{q}) = k$):

$$\nabla U(\mathbf{x}) + \lambda \nabla I(\mathbf{x}) = 0$$

where $\mathbf{x} = (\mathbf{x}_1, \ldots \mathbf{x}_n)$.

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In other words, a c.c. is a critical point of U subject to the constraint $I = k$ (the mass ellipsoid). This gives a useful topological approach to studying central configurations (Smale, Conley, Meyer, McCord, etc.)

A Simple Formula for λ

Recall $U(\mathbf{q}) = \sum_{i<j}^{n} \frac{m_i m_j}{r_{ij}}$ and $I(\mathbf{q}) = \frac{1}{2}\sum_{i=1}^{n} m_i ||\mathbf{q}_i||^2$
Note that U is homogeneous of degree -1 and I is homogeneous of degree 2. Taking the c.c. equation

$$\nabla U(\mathbf{x}) + \lambda \nabla I(\mathbf{x}) = 0$$

and dotting both sides with $\mathbf{x}$ yields (Euler's Theorem for Homogeneous Potentials)

$$-U(\mathbf{x}) + \lambda \cdot 2I(\mathbf{x}) = 0.$$

This gives a simple formula for λ:

$$\lambda = \frac{U(\mathbf{x})}{2I(\mathbf{x})}$$

Homothetic Solutions

Guess a solution of the form $\mathbf{q}_i(t) = r(t)\mathbf{x}_i \; \forall i$ where $\mathbf{x}_i$ is an unknown vector and $r(t)$ an unknown scalar function. Plug it in:

$$
\begin{aligned}
m_i \, \ddot{r} \, \mathbf{x}_i &= \sum_{j \neq i}^{n} \frac{m_i m_j \left(r(t)\mathbf{x}_j - r(t)\mathbf{x}_i \right)}{||r(t)\mathbf{x}_j - r(t)\mathbf{x}_i||^3} \\
&= \frac{r}{|r|^3} \sum_{j \neq i}^{n} \frac{m_i m_j (\mathbf{x}_j - \mathbf{x}_i)}{||\mathbf{x}_j - \mathbf{x}_i||^3} \\
&= \frac{r}{|r|^3} \frac{\partial U}{\partial \mathbf{q}_i}(\mathbf{x})
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$$\begin{aligned} m_i\,\ddot{r}\,\mathbf{x}_i &= \sum_{j\neq i}^{n} \frac{m_i m_j\left(r(t)\mathbf{x}_j - r(t)\mathbf{x}_i\right)}{\|r(t)\mathbf{x}_j - r(t)\mathbf{x}_i\|^3} \\ &= \frac{r}{|r|^3}\sum_{j\neq i}^{n} \frac{m_i m_j(\mathbf{x}_j - \mathbf{x}_i)}{\|\mathbf{x}_j - \mathbf{x}_i\|^3} \\ &= \frac{r}{|r|^3}\,\frac{\partial U}{\partial \mathbf{q}_i}(\mathbf{x}) \end{aligned}$$

Suppose that the $\mathbf{x}_i$'s satisfy $\frac{\partial U}{\partial \mathbf{q}_i}(\mathbf{x}) = -\lambda m_i \mathbf{x}_i$ for each i (ie. they form a central configuration), then $r(t)$ must satisfy

$$\ddot{r} = -\frac{\lambda r}{|r|^3} \qquad \text{1d Kepler problem}$$

Homothetic Solutions (cont.)

The scalar ODE for $r(t)$

$$\ddot{r} = -\frac{\lambda r}{|r|^3}$$

is easily studied (Hamiltonian system) and contains solutions that approach zero in finite time as well as solutions that escape to ∞.

In particular, if $r(0) = r_0$ and $\dot{r}(0) = 0$, then collision, $\lim_{t \to T^-} r(t) = 0$, occurs at time

$$T = \frac{\pi}{\sqrt{\lambda}} \left(\frac{r_0}{2} \right)^{3/2}.$$

One can also check that $r(t) = c\, t^{2/3}$ with $c^3 = 9\lambda/2$ is a solution (parabolic case $h = 0$).

Homographic Solutions

Complexify and guess a solution of the form

$$\mathbf{q}_i(t) = z(t)\,\mathbf{x}_i\;\forall i \qquad \text{with } z(t) : \mathbb{R} \mapsto \mathbb{C}, \mathbf{x}_i \in \mathbb{C}.$$

By similar arguments as with the homothetic case, this leads to

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Therefore, attaching a particular solution of the planar Kepler problem (circular, elliptic, hyperbolic, parabolic) to each body in a planar c.c. yields a solution to the full *n*-body problem.

- Circular Kepler orbit yields a rigid rotation, a relative equilibrium (same shape and size)
- An elliptic Kepler orbit yields a periodic orbit, a relative periodic solution (same shape, oscillating size)

Approaches to Studying CCs

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2. Existence for special cases: For a particular choice of masses (or set of masses), what are the c.c.'s and are there any interesting bifurcations? Success in many cases.

- One large mass, the rest small (forms a ring)
- One small mass, the rest large (a restricted problem). Nice applications to spacecraft transport.
- Equal masses or some other choice of symmetry. Does equal masses imply symmetry in the configuration?
- Almost all equal masses (three out of four equal, two pairs of two equal masses, etc.)

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- Regular n-gon requires equal masses if $n > 3$.
- Nested n-gons (Moeckel and Simó 1995 , Llibre and Mello 2009)
- Pyramid with a square base (Fayçal, 1996)
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4. Generic Results: What properties hold true for all c.c.'s? Existence results.

- For any four masses, there exists at least one convex quadrilateral cc. (MacMillan and Bartky 1932, Xia 2004)
- For any four masses, there exists at least one concave quadrilateral cc. (Hampton, PhD Thesis 2002)
- The Perpendicular Bisector Theorem (Moeckel, 1990)

The Planar, Circular, Restricted 3-Body Problem (PCR3BP)

$\mathbf{q}_1 = (1-\mu, 0), m_1 = \mu$ and $\mathbf{q}_2 = (-\mu, 0), m_2 = 1-\mu \quad (0 < \mu \leq 1/2)$

Let $a = \sqrt{(x-1+\mu)^2 + y^2}, \quad b = \sqrt{(x+\mu)^2 + y^2}$.

Equations of motion for the infinitesimal body (x, y):

$$\begin{aligned} \dot{x} &= u \\ \dot{y} &= v \\ \dot{u} &= V_x + 2v \\ \dot{v} &= V_y - 2u \end{aligned}$$

where

$$V(x,y) = \frac{1}{2}(x^2+y^2) + \frac{\mu}{a} + \frac{1-\mu}{b} + \frac{1}{2}\mu(1-\mu)$$

is the **amended potential.**

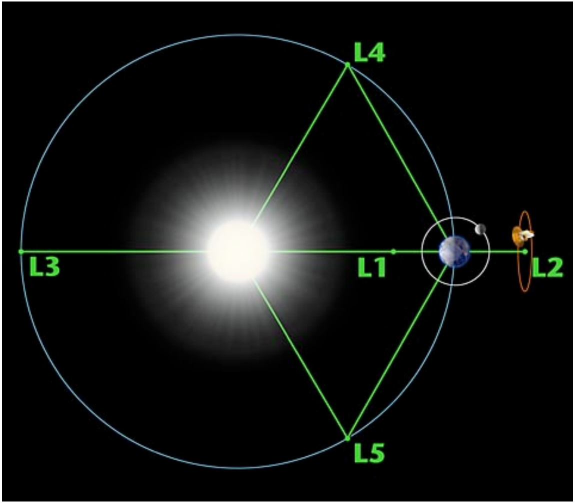

Figure: The five libration points (Lagrange points) in the Sun-Earth system (not drawn to scale).
http://map.gsfc.nasa.gov/mission/observatory_l2.html

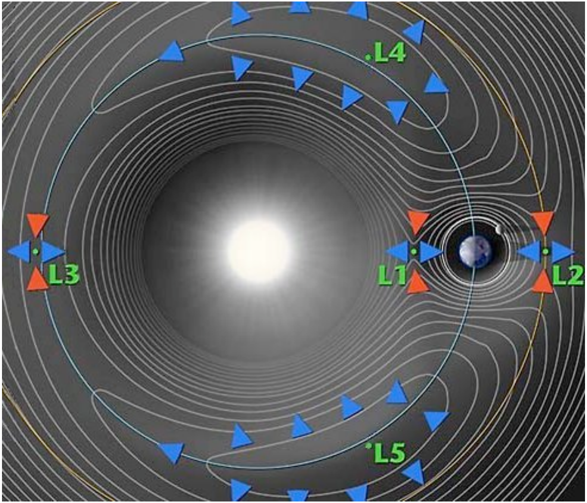

Figure: The level curves for the amended potential and the libration points.
http://map.gsfc.nasa.gov/mission/observatory_l2.html

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5. The L5 Society formed in 1975 to promote the formation of space colonies at the L_4 or L_5 points in the Earth-Moon system. From the first newsletter: "our clearly stated long range goal will be to disband the Society in a mass meeting at L5."

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4. Lots of examples in Science Fiction: In the Star Trek: The Next Generation episode, "The Survivors", the Enterprise is surprised by an enemy ship that had been hiding in a Lagrange point.

RICHARD MOECKEL AND CARLES SIMÓ

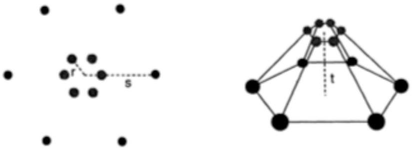


FIG. 1. *The symmetrical configurations considered here.*

Moeckel and Simó, "Bifurcation of spatial central configurations from planar ones"

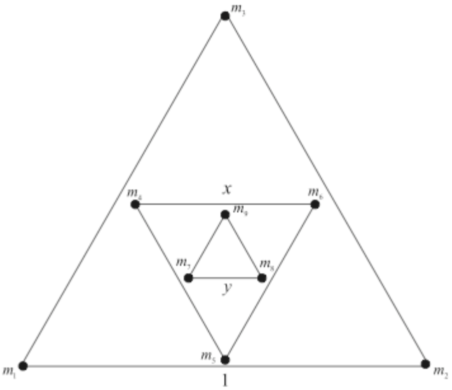


FIGURE 1. Three nested triangular central configurations.

Llibre and Mello, "Triple and quadruple nested central configurations for the planar *n*-body problem," 2009

JAUME LLIBRE AND LUIS FERNANDO MELLO

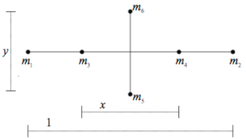


FIGURE 2. Three nested 2–collinear central configurations.

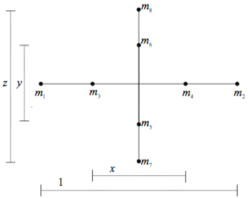


FIGURE 3. Four nested 2–collinear central configurations.

Llibre and Mello, "Triple and quadruple nested central configurations for the planar n-body problem," 2009

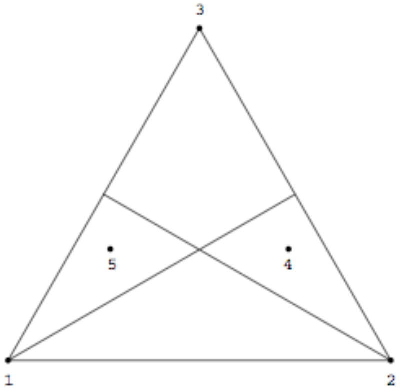


Figure 1. A typical configuration.

Hampton, "Stacked central configurations: new examples in the planar five-body problem," 2005

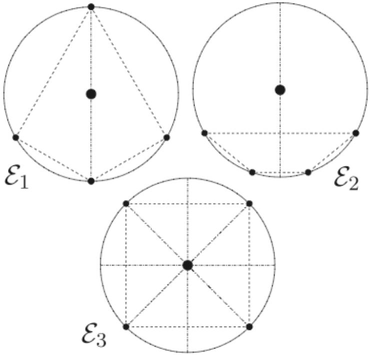


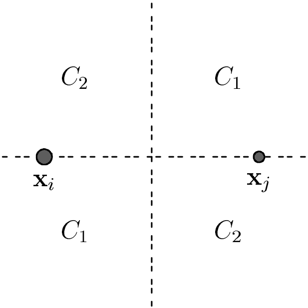
Figure 1: The three relative equilibria of 4 separate identical satellites

Albouy and Fu, "Relative equilibria of four identical satellites," 2009

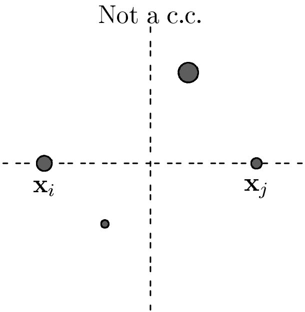
The Perpendicular Bisector Theorem (Moeckel, 1990)

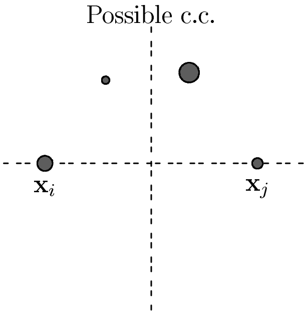
Theorem

Suppose that $\mathbf{x}$ *is a planar c.c. and let* $\mathbf{x}_i$ *and* $\mathbf{x}_j$ *be any two of its points. Then, if one of the two open cones determined by the line through* $\mathbf{x}_i$ *and* $\mathbf{x}_j$ *and its perpendicular bisector contain points of the configuration, so must the other one.*

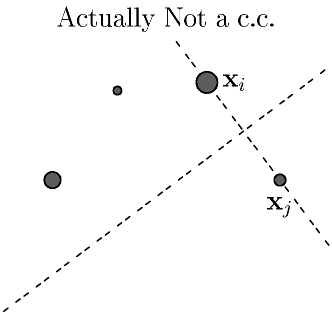


Perpendicular Bisector Thm. — Examples





Perpendicular Bisector Thm. — Examples (Cont.)



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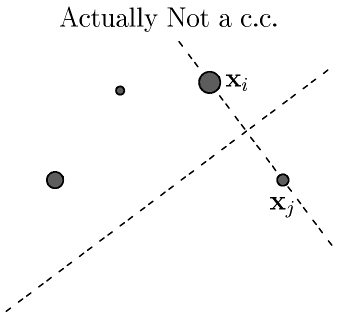

Corollary

The only possible non-collinear three-body central configuration is the equilateral triangle.

Collinear Central Configurations

Set $d = 1$, so $\mathbf{q} = (q_1, \ldots, q_n) \in \mathbb{R}^n$. Let $\Delta' = \{\mathbf{q} \in \mathbb{R}^n : q_i = q_j \text{ for some } i \neq j\}$ (collision set). The configuration space for the collinear n-body problem is $\mathbb{R}^n - \Delta'$.

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Set $S' = \{\mathbf{q} \in \mathbb{R}^n : I(\mathbf{q}) = 1\}$ (mass ellipsoid) and $P = \{\mathbf{q} \in \mathbb{R}^n : \sum_i m_i q_i = 0\}$ (center of mass at origin).

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Example $n = 3$: S' is a 2-sphere, P is a plane and S is a great circle. The collision set Δ' intersects S in 6 points, one for each ordering of q_1, q_2, q_3.

Collinear Central Configurations (Cont.)

Let $\Delta = S \cap \Delta'$ be the intersection of the collision planes with S. Topologically, Δ contains spheres of dimension $n-3$.

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Let V be the restriction of U to $S - \Delta$. A critical point of V is a c.c. Since $\lim\limits_{\mathbf{q} \to \Delta} V = \infty$, there is at least one minimum per connected component.

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One can show that the Hessian is always positive definite at any critical point, (concave up), thus the only critical points are minima and there are precisely $n!$ of them, one for each ordering of the variables. (Moulton, 1910, *Annals of Mathematics*)

Degeneracies and Counting

$$\sum_{j \neq i} \frac{m_i m_j (\mathbf{x}_j - \mathbf{x}_i)}{r_{ij}^3} + \lambda m_i \mathbf{x}_i = 0, \quad i = \{1, 2, \ldots, n\} \tag{1}$$

$\mathbf{x} = (\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n)$ is a c.c. implies that

$$c\mathbf{x} = (c\mathbf{x}_1, c\mathbf{x}_2, \ldots, c\mathbf{x}_n) \quad \text{and}$$

$$R\mathbf{x} = (R\mathbf{x}_1, R\mathbf{x}_2, \ldots, R\mathbf{x}_n)$$

are also c.c.'s, where c is a scalar and $R \in SO(d)$.

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are also c.c.'s, where c is a scalar and $R \in SO(d)$.

Let S be the ellipsoid defined by $2I = 1$ (fixes scaling). Define an equivalence relation via $\mathbf{x} \sim R\mathbf{x}, \; R \in SO(d)$ (identify configurations equivalent under a rotation).

Critical points of $U([\mathbf{x}])$ on $S/\sim$ are central configurations. When counting c.c.'s, one usually counts equivalence classes.

Finiteness

The Smale/Wintner/Chazy Question: For a fixed choice of masses, is the number of equivalence classes of planar central configurations finite? (Smale's 6th problem for the 21st century)

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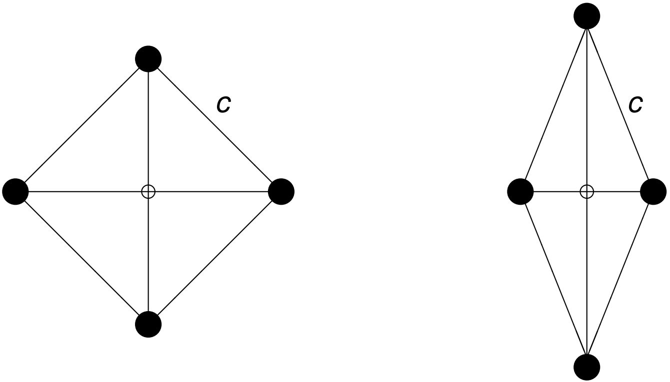
- $n \geq 5$ Open problem!
- A monkey wrench: There exists a continuum of c.c.'s in the planar 5-body problem with masses $m_1 = m_2 = m_3 = m_4 = 1$ and $m_5 = -1/4$ (GR, 1999)

The 1+rhombus one-parameter family of c.c.'s



Bodies on rhombus have mass 1 while "body" at center has mass $-1/4$. Configuration is a c.c. as long as the side length of the rhombus stays constant (interior angle serves as a parameter, $\lambda = 2/c^3$).

The Planar, Circular, Restricted Four-Body Problem (PCR4BP)

Take three masses ("primaries") on a Lagrange equilateral triangle relative equilibrium and insert a fourth infinitesimal mass that has no influence on the circular orbits of the larger bodies. Change to a rotating coordinate system in a frame where the primaries are fixed. Let (x, y) be coordinates for the infinitesimal mass in this new frame.

Equations of motion: (assume $m_1 + m_2 + m_3 = 1$)

$$\begin{aligned} \ddot{x} &= 2\dot{y} + V_x \\ \ddot{y} &= -2\dot{x} + V_y \end{aligned}$$

where

$$V(x,y) = \frac{1}{2}\left((x - c_x)^2 + (y - c_y)^2\right) + \frac{m_1}{a} + \frac{m_2}{b} + \frac{m_3}{c}$$

is the amended potential, (c_x, c_y) is the center of mass of the primaries and a, b, c represent the respective distances of the infinitesimal mass from each of the three primaries.

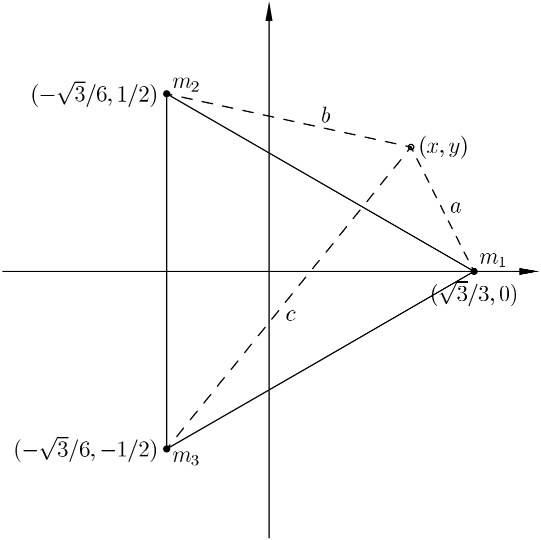


Figure: Setup for the planar, circular, restricted four-body problem.

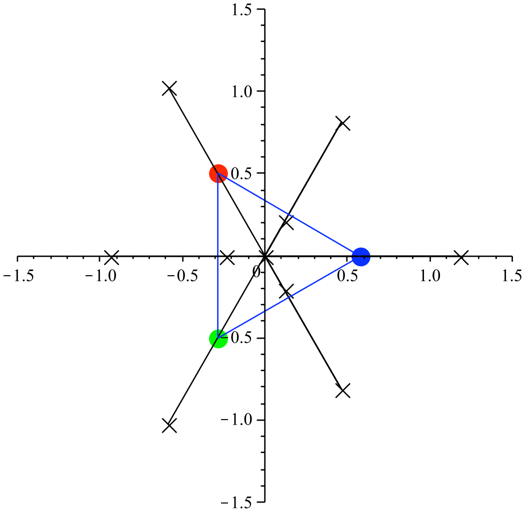


Figure: The 10 equilibria for the PCR4BP in the case of equal masses. Note the symmetry with respect to the equilateral triangle of the primaries.

Theorem

(Kulevich, GR, Smith 2008) The number of equilibria in the PCR4BP is finite for any choice of masses. In particular, there are at most 196 critical points.

Remarks:

1. Our result showing finiteness appears to be new. Leandro (2006) uses linear fractional transformations and resultants to prove that no bifurcations occur in the number of critical points *outside* the triangle of primaries, thus giving an exact count of 6 equilibria outside the triangle of primaries.
2. The upper bound of 196 is clearly not optimal as the work of Pedersen (1944), Simó (1978), Arenstorf (1982) and Leandro (2006) suggests that the actual number varies between 8 and 10. It is a surprisingly complicated problem to study the bifurcation curve in the mass parameter space for which there are precisely 9 critical points.

Mutual Distances Make Great Coordinates

Recall:

$$U(\mathbf{q}) = \sum_{i<j}^{n} \frac{m_i m_j}{r_{ij}}$$

Alternative formula for I in terms of mutual distances: (center of mass at origin)

$$I(\mathbf{q}) = \frac{1}{2M} \sum_{i<j}^{n} m_i m_j r_{ij}^2$$

where $M = m_1 + \cdots + m_n$ is the total mass.

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Key Observation: The Smale/Wintner/Chazy question can be formulated using the mutual distances as coordinates. The problem can be reduced to showing a system of polynomial equations has a finite number of solutions.

The Lagrange Equilateral Triangle Solution

Suppose $n = 3$ and we seek only planar c.c.'s. Since we are identifying triangles identical under a translation and/or rotation, the three mutual distances r_{12}, r_{13}, r_{23} serve as independent coordinates by the SSS Postulate of Euclidean geometry.

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$$-\frac{m_i m_j}{r_{ij}^2} + \lambda \frac{m_i m_j}{M} r_{ij} = 0$$

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This is true for any pair (i, j) so that all mutual distances must be equal to the same constant! Thus the equilateral triangle is the only non-collinear c.c. in the 3-body problem.

Generalizing Lagrange

In the four-body problem in $\mathbb{R}^3$, the six mutual distances $r_{12}, r_{13}, r_{14}, r_{23}, r_{24}, r_{34}$ specify a unique tetrahedron up to translation and rotation. Again, these coordinates can be used as variables.

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Solution: $r_{ij} = (M/\lambda)^{1/3}$

This is true for any pair (i, j) so that all mutual distances are equal. Thus the regular tetrahedron is the only non-planar c.c. in the four-body problem.

Generalizing Lagrange (cont.)

Theorem

The regular $n-1$ dimensional simplex with n arbitrary masses is a central configuration for the n-body problem. It is the only c.c. of this dimension.

Generalizing Lagrange (cont.)

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The regular $n-1$ dimensional simplex with n arbitrary masses is a central configuration for the n-body problem. It is the only c.c. of this dimension.

In order to find the lower dimensional c.c.'s we must add further restrictions on the mutual distances. For example, to find collinear c.c.'s in the 3-body problem, we could require

$$F \;=\; r_{12} + r_{23} - r_{13} \;=\; 0$$

as an additional constraint. In other words, the collinear c.c.'s for the ordering $q_1 < q_2 < q_3$ would be critical points of U subject to the constraints $I = k$ and $F = 0$.

4-body Planar CC's

To use the six mutual distances $r_{12}, r_{13}, r_{14}, r_{23}, r_{24}, r_{34}$ as variables, we need an additional constraint that ensures the configuration is planar. We require that the volume of the tetrahedron be zero (Cayley-Menger determinant).

$$F = \begin{vmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & r_{12}^2 & r_{13}^2 & r_{14}^2 \\ 1 & r_{12}^2 & 0 & r_{23}^2 & r_{24}^2 \\ 1 & r_{13}^2 & r_{23}^2 & 0 & r_{34}^2 \\ 1 & r_{14}^2 & r_{24}^2 & r_{34}^2 & 0 \end{vmatrix} = 0$$

4-body Planar CC's

To use the six mutual distances $r_{12}, r_{13}, r_{14}, r_{23}, r_{24}, r_{34}$ as variables, we need an additional constraint that ensures the configuration is planar. We require that the volume of the tetrahedron be zero (Cayley-Menger determinant).

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Goal: Find critical points of $U + \lambda(I - k) + \frac{\sigma}{64} F$ (two Lagrange multipliers) satisfying $I = k$ and $F = 0$.

4-body Planar CC's cont.

Amazing fact: When $F = 0$,

$$\frac{\partial F}{\partial r_{ij}^2} = 32 A_i A_j$$

where A_i is the oriented area of the triangle not including $\mathbf{x}_i$. For example, A_1 is the oriented area of the triangle formed by bodies $\mathbf{x}_2, \mathbf{x}_3$ and $\mathbf{x}_4$.

4-body Planar CC's cont.

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Differentiating w.r.t. r_{ij}^2 leads to six equations of the form:

$$m_i m_j (\lambda' - r_{ij}^{-3}) + \sigma A_i A_j = 0$$

where $\lambda' = \lambda / M$.

Dziobek's Equations

$$m_1 m_2(r_{12}^{-3} - \lambda') = \sigma A_1 A_2 \qquad m_3 m_4(r_{34}^{-3} - \lambda') = \sigma A_3 A_4$$

$$m_1 m_3(r_{13}^{-3} - \lambda') = \sigma A_1 A_3 \qquad m_2 m_4(r_{24}^{-3} - \lambda') = \sigma A_2 A_4$$

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Multiply pairwise gives identical right-hand sides! This leads to a famous set of equations, first discovered by Dziobek (1900).

$$(r_{12}^{-3} - \lambda')(r_{34}^{-3} - \lambda') = (r_{13}^{-3} - \lambda')(r_{24}^{-3} - \lambda') = (r_{14}^{-3} - \lambda')(r_{23}^{-3} - \lambda')$$

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Necessary and Sufficient: If these last equations are satisfied for a planar configuration, then the ratios of the masses can be obtained by dividing appropriate pairs in the first list. However, positivity of the masses must still be checked.

Convex 4-body CC's

A sample ratio:

$$\frac{m_1 A_2}{m_2 A_1} = \frac{r_{23}^{-3} - \lambda'}{r_{13}^{-3} - \lambda'} = \frac{r_{24}^{-3} - \lambda'}{r_{14}^{-3} - \lambda'} = \frac{r_{23}^{-3} - r_{24}^{-3}}{r_{13}^{-3} - r_{14}^{-3}}$$

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Requiring positivity of the masses enforces the following requirements on the mutual distances in the convex case:

- The diagonals must be longer than all exterior sides.
- The longest and shortest exterior sides are opposite each other. (Thus, the only possible rectangle is a square.)
- The ratio of the lengths of the diagonals must lie between $1/\sqrt{3}$ and $\sqrt{3}$.
- The size of the interior angles must be between 30° and 120°.

Some Approachable Problems?

- Find all central configurations in the four-body problem lying on a circle (co-circular c.c.'s). If the center of mass coincides with the center of the circle, the only possibility is the square with equal masses (Hampton, 2003). This is related to a problem posed by Alain Chenciner: Are there any perverse choreographies? A choreography (all bodies trace out the same curve) is perverse if it is a solution to the n-body problem for more than one set of masses (not scaling).

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- What is the connection between symmetry in the masses and symmetry in the corresponding central configuration? For example, all c.c.'s for four equal masses have a line of symmetry. Numerically this looks to be true for $n = 5, 6, 7$ but not $n = 8$ (Moeckel). Can this be proven? What about two pairs of equal masses in the four-body problem?

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- Finiteness: Smale/Wintner/Chazy $n = 5$, restricted problems (PCR5BP)