Cyclic Central Configurations in the Four-Body Problem

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Definition

A *central configuration* (c.c.) is a configuration of bodies $(\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n), \mathbf{x}_i \in \mathbb{R}^d$ such that the acceleration vector for each body is a common scalar multiple of its position vector. Specifically, in the Newtonian n-body problem with the center of mass at the origin, for each index i,

$$\sum_{j \neq i}^{n} \frac{m_i m_j (\mathbf{x}_j - \mathbf{x}_i)}{||\mathbf{x}_j - \mathbf{x}_i||^3} + \lambda\, m_i \mathbf{x}_i = 0$$

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- The collinear c.c.'s correspond to $d = 1$, planar c.c.'s to $d = 2$, spatial c.c.'s to $d = 3$. One can also study theoretically the case $d > 3$.
- Summing together the n equations above quickly yields $\sum_i m_i \mathbf{x}_i = 0$.

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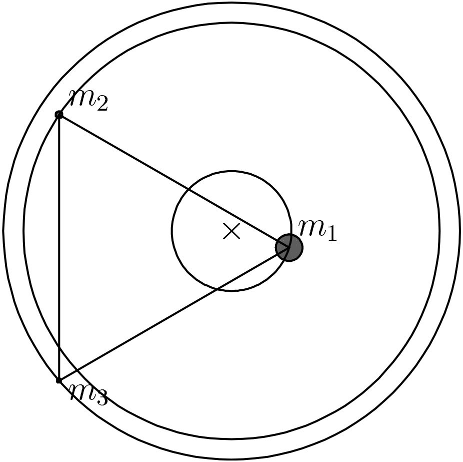
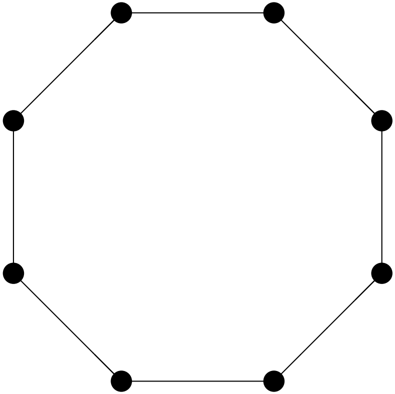
- 209 articles found on MathSciNet using a general search for "central configurations"

Equilateral Triangle (Lagrange 1772)



Regular n-gon (equal mass required for $n \geq 4$)



Cyclic CC's

Goal: Study the set of 4-body planar, convex, central configurations lying on a circle. Such a configuration will be called a cyclic central configuration (c.c.c) since the quadrilateral formed by the positions of the four bodies is cyclic.

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- If the center of mass coincides with the center of the circle, the only possibility is the square with equal masses (Hampton, 2003). If such a co-circular c.c. existed for more bodies, other than the regular n-gon, this would be a choreography (all bodies tracing out the same curve) **without** equal time spacing between bodies.

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- Spatial 5-body pyramidal c.c.'s exist where four bodies lie on a sphere with the fifth body at the center of the sphere. The four bodies forming the base of the configuration are co-circular and form a c.c.c. in the 4-body problem (Fayçal, 1996; Albouy, 2003)

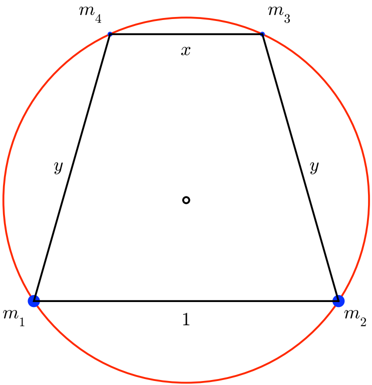

Figure: An example of a cyclic central configuration. The center of the circumscribing circle is marked with an O.

Let $r_{ij} = ||\mathbf{q}_i - \mathbf{q}_j||$ where $\mathbf{q}_i$ denotes the position of the i-th body. The Newtonian potential function is

$$U(\mathbf{q}) = \sum_{i<j}^{n} \frac{m_i m_j}{r_{ij}}$$

The equations of motion for the n-body problem are then given by

$$\begin{aligned} m_i \ddot{\mathbf{q}}_i &= \frac{\partial U}{\partial \mathbf{q}_i}, \qquad i \in \{1, 2, \ldots n\} \\ &= \sum_{j \neq i}^{n} \frac{m_i m_j (\mathbf{q}_j - \mathbf{q}_i)}{r_{ij}^3} \end{aligned}$$

An Alternate Characterization of CC's

Let $r_{ij} = ||\mathbf{q}_i - \mathbf{q}_j||$ where $\mathbf{q}_i$ denotes the position of the i-th body. The Newtonian potential function is

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Consequently, the i-th equation for a c.c. can be written as

$$\frac{\partial U}{\partial \mathbf{q}_i}(\mathbf{x}) + \lambda m_i \mathbf{x}_i = 0.$$

CC's as critical points of U

The moment of inertia $I(\mathbf{q})$ (w.r.t. the center of mass) is defined as

$$I(\mathbf{q}) = \frac{1}{2} \sum_{i=1}^{n} m_i ||\mathbf{q}_i||^2.$$

Thus, the equations for a c.c. can be viewed as a Lagrange multiplier problem (set $I(\mathbf{q}) = k$):

$$\nabla U(\mathbf{x}) + \lambda \nabla I(\mathbf{x}) = 0$$

where $\mathbf{x} = (\mathbf{x}_1, \ldots \mathbf{x}_n)$.

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In other words, a c.c. is a critical point of U subject to the constraint $I = k$ (the mass ellipsoid). This gives a useful topological approach to studying central configurations (Smale, Conley, Meyer, McCord, etc.)

Mutual Distances Make Great Coordinates

Recall:

$$U(\mathbf{q}) = \sum_{i<j}^{4} \frac{m_i m_j}{r_{ij}}$$

Alternative formula for I in terms of mutual distances: (center of mass at origin)

$$I(\mathbf{q}) = \frac{1}{2M} \sum_{i<j}^{4} m_i m_j r_{ij}^2$$

where $M = m_1 + \cdots + m_4$ is the total mass.

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Problem: The six variables $r_{12}, r_{13}, r_{14}, r_{23}, r_{24}$ and r_{34} are not independent in the planar problem. Generically, they describe a tetrahedron, not a planar configuration. It is easy to see that the regular tetrahedron is the only non-planar c.c. in the four-body problem.

4-body Planar CC's

To use the six mutual distances $r_{12}, r_{13}, r_{14}, r_{23}, r_{24}, r_{34}$ as variables, we need an additional constraint that ensures the configuration is planar. We require that the volume of the tetrahedron be zero (Cayley-Menger determinant).

$$F = \begin{vmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & r_{12}^2 & r_{13}^2 & r_{14}^2 \\ 1 & r_{12}^2 & 0 & r_{23}^2 & r_{24}^2 \\ 1 & r_{13}^2 & r_{23}^2 & 0 & r_{34}^2 \\ 1 & r_{14}^2 & r_{24}^2 & r_{34}^2 & 0 \end{vmatrix} = 0$$

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Cumbersome! Is there a simpler constraint that ensures both planarity and that the bodies lie on a circle?

Ptolemy's Theorem

If four bodies lie on a common circle and are numbered sequentially (ie. the diagonals have lengths r_{13} and r_{24}), then $P = 0$, where

$$P = r_{12}r_{34} + r_{14}r_{23} - r_{13}r_{24}.$$

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Theorem (Apostol, 1967): For any convex quadrilateral numbered sequentially or for any tetrahedron, $P \geq 0$ with equality iff the four bodies lie on a circle.

If we restrict to the space of geometrically realizable configurations, $r_{ij} > 0$ and $r_{ij} + r_{jk} > r_{ik}$ for all possible triples of indices (i, j, k), then $P = 0$ iff the configuration is cyclic (with a sequential ordering.)

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Goal: Find critical points of $U + \lambda M(I - I_0) + \sigma P$ satisfying $I = I_0$ and $P = 0$.

Using the six mutual distances as variables, we find

$$m_1 m_2(r_{12}^{-3} - \lambda) = \sigma \frac{r_{34}}{r_{12}}, \qquad m_3 m_4(r_{34}^{-3} - \lambda) = \sigma \frac{r_{12}}{r_{34}}$$

$$m_1 m_3(r_{13}^{-3} - \lambda) = -\sigma \frac{r_{24}}{r_{13}}, \qquad m_2 m_4(r_{24}^{-3} - \lambda) = -\sigma \frac{r_{13}}{r_{24}}$$

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This yields a well-known relation of Dziobek (1900)

$$(r_{12}^{-3} - \lambda)(r_{34}^{-3} - \lambda) = (r_{13}^{-3} - \lambda)(r_{24}^{-3} - \lambda) = (r_{14}^{-3} - \lambda)(r_{23}^{-3} - \lambda) \qquad (1)$$

which must be true for any planar 4-body c.c., not just cyclic c.c.'s.

Eliminating λ from equation (1) in a clever way yields

$$(r_{13}^3 - r_{12}^3)(r_{23}^3 - r_{34}^3)(r_{24}^3 - r_{14}^3) \;=\; (r_{12}^3 - r_{14}^3)(r_{24}^3 - r_{34}^3)(r_{13}^3 - r_{23}^3). \quad (2)$$

Equation (2) is necessary and sufficient for a 4-body planar c.c. given that the six mutual distances determine a geometrically realizable planar configuration. However, it does *not* ensure positivity of the masses.

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Generically, the space of 4-body planar c.c.'s is three dimensional since it is described by the equations $F = 0, I = I_0$ and equation (2). Restricting to a circle yields a two dimensional space.

Mass Ratios

$$\frac{m_2}{m_1} = \frac{(\lambda - r_{13}^{-3})\, r_{13} r_{14}}{(r_{23}^{-3} - \lambda)\, r_{23} r_{24}} = \frac{(r_{14}^{-3} - \lambda)\, r_{13} r_{14}}{(\lambda - r_{24}^{-3})\, r_{23} r_{24}}$$

$$\frac{m_3}{m_1} = \frac{(r_{12}^{-3} - \lambda)\, r_{12} r_{14}}{(r_{23}^{-3} - \lambda)\, r_{23} r_{34}} = \frac{(r_{14}^{-3} - \lambda)\, r_{12} r_{14}}{(r_{34}^{-3} - \lambda)\, r_{23} r_{34}}$$

$$\frac{m_4}{m_1} = \frac{(r_{12}^{-3} - \lambda)\, r_{12} r_{13}}{(\lambda - r_{24}^{-3})\, r_{24} r_{34}} = \frac{(\lambda - r_{13}^{-3})\, r_{12} r_{13}}{(r_{34}^{-3} - \lambda)\, r_{24} r_{34}}.$$

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Without loss of generality, let $r_{12} = 1$ be the longest exterior side and let $r_{14} \geq r_{23}$. Requiring positive masses gives

$$r_{13} \geq r_{24} > r_{12} = 1 \geq r_{14} \geq r_{23} \geq r_{34}. \tag{3}$$

The diagonals are longer than any of the exterior sides. The longest exterior side is opposite the smallest.

A quadrilateral is a kite if two opposite bodies lie on an axis of symmetry of the configuration. Equation (2) is immediately satisfied when $r_{12} = r_{14} = 1$ and $r_{23} = r_{34}$, yielding a kite configuration with an axis of symmetry between bodies 1 and 3.

Set $r_{23} = r_{34} = x$ and the diagonals $r_{13} = c$ and $r_{24} = 2x/c$ where $x \in (0, 1]$ is a parameter and $c = \sqrt{1 + x^2}$. These distances satisfy Ptolemy's relation $P = 0$.

We find $m_2 = m_4$, as expected from symmetry. Setting $m_1 = 1$, we have $m_2 = m_4 = m$ and $m_3 = \alpha m$ where

$$m = \frac{4x(c^3 - 1)}{c(8 - c^3)} \quad \text{and} \quad \alpha = \frac{c(8x^3 - c^3)}{4(c^3 - x^3)}.$$

To ensure positive masses, we must have $1/\sqrt{3} < x \leq 1$.

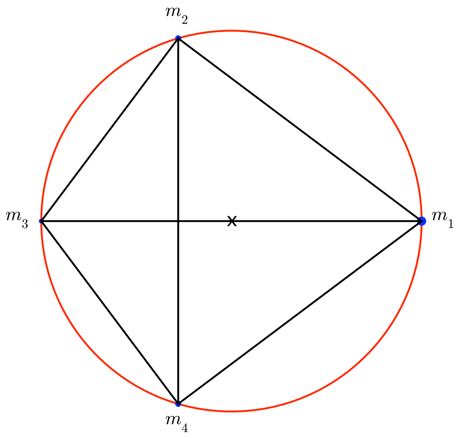

Figure: An example of a cyclic kite central configuration with $m_2 = m_4$.

Theorem

There exists a one-parameter family of cyclic kite central configurations with bodies one and three lying on the diameter of the circumscribing circle. The masses are $m_1 = 1, m_2 = m_4 = m$ *and* $m_3 = \alpha\, m$ *and are ordered* $m_1 \geq m_2 = m_4 \geq m_3$ *with equality iff the configuration is a square. At one end of the family* ($x = 1/\sqrt{3}$) *is a c.c. of the planar, restricted 4-body problem, with bodies 1, 2 and 4 forming an equilateral triangle and* $m_3 = 0$. *At the other end* ($x = 1$) *is the square with equal masses.*

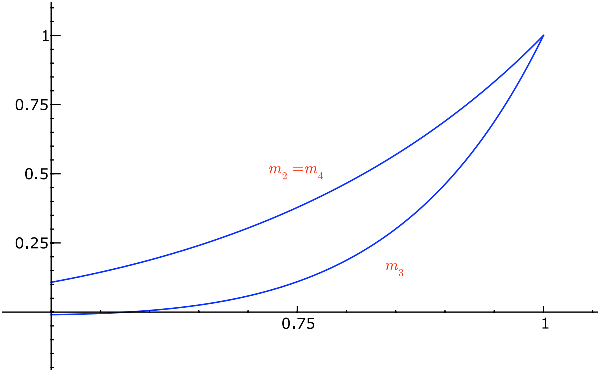

Figure: The values of the masses for the cyclic kite c.c.'s

The longest arc along the cirmcumscribing circle, denoted θ_{12}, occurs between bodies 1 and 2. We have that

$$\theta_{12} = \pi - 2\arctan(x)$$

and the maximum arc is 120° ($m_3 = 0$) while the minimum arc is 90° (square).

Theorem

The supremum of the largest interior angle of the kite ccc is 120° *while the infimum of the smallest interior angle is* 60°*. Moreover, the arc length along the circumscribing circle between bodies 1 and 2 is a decreasing function of the parameter* x *with a supremum of* $2\pi/(3\sqrt{3}) \approx 1.2092$ *and a minimum attained at the square configuration of* $2^{-3/2}\pi \approx 1.1107$*.*

Symmetric Example: Isosceles Trapezoid

Theorem

A cyclic central configuration is an isosceles trapezoid if and only if $m_1 = m_2$ *and* $m_3 = m_4$.

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Key calculation in proof: If $m_1 = m_2$, then

$$r_{23}^2 r_{24}^2 (r_{13}^3 - r_{14}^3) - r_{13}^2 r_{14}^2 (r_{24}^3 - r_{23}^3) \; = \; 0. \tag{4}$$

Likewise, if $m_3 = m_4$, then

$$r_{13}^2 r_{23}^2 (r_{24}^3 - r_{14}^3) - r_{14}^2 r_{24}^2 (r_{13}^3 - r_{23}^3) \; = \; 0. \tag{5}$$

Taking the difference of equations (4) and (5) and factoring the result yields

$$(r_{13} - r_{24})(r_{13}^2 r_{14}^2 r_{24}^2 + r_{13}^2 r_{23}^2 r_{24}^2 + r_{13} r_{14}^3 r_{23}^2 + r_{13} r_{14}^2 r_{23}^3 + r_{14}^3 r_{23}^2 r_{24} + r_{14}^2 r_{23}^3 r_{24}) = 0.$$

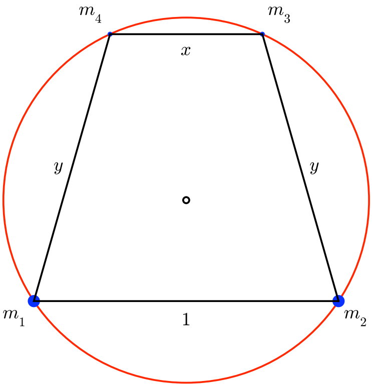

Figure: An example of an isosceles trapezoid central configuration. The center of the circumscribing circle is marked with an O.

We set $m_1 = m_2 = 1$. Then, $m_3 = m_4 = m$ where

$$m \;=\; \frac{x^2(1-y^3)}{y^3-x^3}.$$

In order for the isosceles trapezoid to be a c.c., we must have

$$f \;=\; (y^2+x)^{3/2}(2y^3-x^3-1) - y^3 - x^3y^3 + 2x^3 \;=\; 0. \qquad (6)$$

Examining $f = 0$, we see that while the smallest side of the trapezoid (parallel to the base) can range from 0 to 1, the length of the congruent legs is considerably constrained between approximately 0.91 and 1.

Conjecture: y is a differentiable function of x and m is an increasing function of x ranging from 0 to 1.

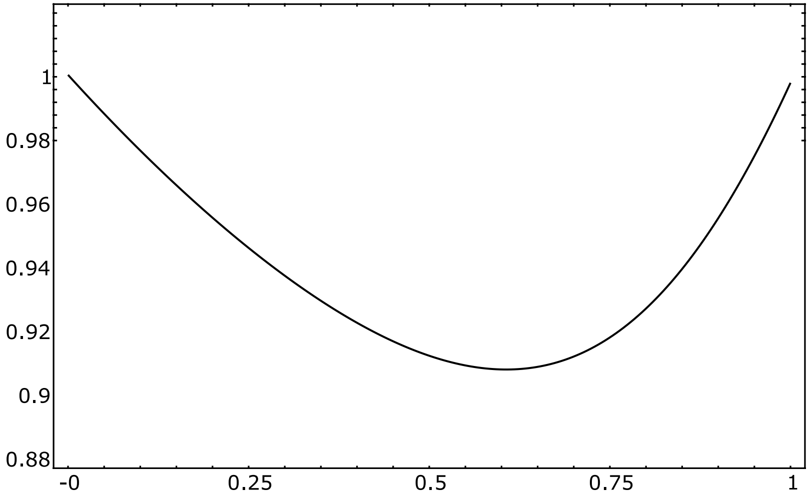

Figure: The relationship between the two distances $x = r_{34}$ and $y = r_{14} = r_{23}$ in the isosceles trapezoid family of cc's.

Theorem

The maximum arclength θ_{12} *in the isosceles trapezoid family of cc's is a monotonically decreasing function of the smallest sidelength* x. *The maximum of* θ_{12} *is* 120° *at the equilateral triangle* ($m_3 = m_4 = 0$ *and* $r_{34} = 0$) *while the minimum is* 90° *attained at the square configuration.*

Note that these bounds are the same as those for the kite family of cyclic central configurations.

Some Geometric Properties of CCC's

1. The maximum interior angle of a 4-body cyclic central configuration lies between 90° and 120° while the minimum interior angle lies between 60° and 90°.
2. Any cyclic c.c. containing two opposite bodies (1 opposite 3 or 2 opposite 4) on a diameter of the circumscribing circle must be a kite configuration.
3. No cyclic c.c. can lie entirely in a semi-circle.
4. Three bodies of a c.c.c. cannot lie on the same half of the circle as the longest side r_{12} unless the configuration is a kite.

Some Open Questions/Future Work

For a cyclic quadrilateral ordered sequentially, the lengths of each diagonal r_{13} and r_{24} can be written as

$$r_{13} = \sqrt{\frac{ab}{c}} \quad \text{and} \quad r_{24} = \sqrt{\frac{ac}{b}} \tag{7}$$

where $a = r_{12}r_{34} + r_{14}r_{23}$, $b = r_{12}r_{14} + r_{23}r_{34}$ and $c = r_{12}r_{23} + r_{14}r_{34}$.

If the two equations in (7) hold, then both $P = 0$ and $V = 0$ follow. Substituting formulae (7) for r_{13} and r_{24} into

$$(r_{13}^3 - r_{12}^3)(r_{23}^3 - r_{34}^3)(r_{24}^3 - r_{14}^3) \;=\; (r_{12}^3 - r_{14}^3)(r_{24}^3 - r_{34}^3)(r_{13}^3 - r_{23}^3)$$

and setting $r_{12} = 1$ yields a complicated equation in three variables,

$$F(r_{14}, r_{23}, r_{34}) \;=\; 0.$$

Let Ω denote the set of mutual distances satisfying

$$r_{13} \ \geq \ r_{24} \ > \ r_{12} = 1 \ \geq \ r_{14} \ \geq r_{23} \ \geq \ r_{34}$$

and $\pi(\Omega)$ its projection onto $r_{14} r_{23} r_{34}$ space.

Let Γ denote the intersection of $F = 0$ and $\pi(\Omega)$. Any point on the surface Γ yields a c.c.c. with positive masses.

Let $G : \Gamma \mapsto \mathbb{R}^3$ be the mass map taking a point in Γ to a 3-tuple of positive mass ratios.

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Key Question: Is G one-to-one? In other words, if the four masses are fixed and a cyclic c.c. exists, is it unique?

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What do the surfaces $F = 0$ and Γ look like? What does the image of G look like in the mass space? The boundaries are determined by the kite and isosceles trapezoid configurations.