Research Overview

Celestial Mechanics

Much of my research is in the field of celestial mechanics, a branch of mathematics that incorporates ideas from differential equations, dynamical systems, algebraic geometry, physics and many other fields. The subject of celestial mechanics, otherwise known as the Newtonian n-body problem, concerns the motion of n celestial bodies (stars, planets, asteroids, spaceships, etc.) interacting solely under gravity. Here, n is a whole number larger than one. The motion of the entire system is described by a complicated set of nonlinear differential equations and is thus an inherently mathematical subject.

The problem is well-understood for the case $n = 2$, a two-body problem such as the sun and the Earth, often referred to as the *Kepler problem*. In this case, a small mass orbits a larger body, its trajectory tracing out the path of a conic section (circle, ellipse, parabola or hyperbola). For over three-hundred years, mathematicians and scientists have strived to understand the complicated structure of the n-body problem for n at least three, although at times with little or moderate success. For example, in attempting to explain known anomalies in the motion of the Earth's moon, Newton investigated the three-body problem by incorporating the gravitational effects of the sun, but was so distraught he confided to a colleague that "his head never ached but with his study on the moon" [19]. On the other hand, research in the n-body problem has enabled us to fly to the moon, place satellites in orbit around the Earth, design affordable spacecraft missions to explore the solar system and locate interesting celestial orbits [5].

My research in celestial mechanics has focused on three particular areas, the stability of periodic solutions, the existence of relative equilibria and Saari's conjecture. Two significant types of periodic solutions that I have studied are *relative equilibria* and *choreographies*. A solution where the bodies are rigidly rotating about their center of mass is called a relative equilibrium, for it appears fixed in a rotating coordinate frame. (Imagine rotating along with the bodies at the same speed, as on a celestial record player. From your perspective, the bodies will appear fixed.) For example, three masses of any size, placed at the vertices of an equilateral triangle and given the correct initial velocities, describe a famous relative equilibrium solution of the three-body problem discovered by Lagrange in 1772. This periodic solution is a rotating equilateral triangle configuration that maintains its shape and size.

In contrast, a solution for which n bodies chase each other around the same curve, with equally spaced time intervals, is called a choreography. Other than the rather trivial circular choreographies (such as the Lagrange solution with equal masses) these types of solutions were only discovered relatively recently. In 1999, Chenciner and Montgomery proved the existence of a remarkable figure-eight choreography in the 3-body problem [3]. The highly symmetric orbit consists of three equal masses traversing a fixed figure-eight loop in a plane. Although a numerical argument had been put forward in 1993 by the physicist Moore [8], it was Chenciner and Montgomery who were able to apply variational methods to prove rigorously the existence of such an orbit. They accomplish this by showing that the figure-eight minimizes the action over a suitable class of curves satisfying certain symmetry constraints. This was the first example of a non-trivial choreography in the n-body problem and its discovery led to hundreds more such solutions, although most of these were found numerically [4].

The stability of these two types of periodic solutions in the n-body problem has important applications to spacecraft trajectory and to understanding the behavior of nearby orbits. In physical terms, stable solutions are the kind we can expect to actually "see" in the universe. A small perturbation (eg. a passing comet or asteroid) does not break up a stable periodic solution. In mathematical terms, studying stability involves computing the Floquet multipliers of the corresponding orbit and showing they lie on the unit circle. This is rather straight-forward conceptually but analytically quite challenging to accomplish.

One of the remarkable features of the figure-eight orbit is its stability. Numerical calculations by Simó have shown that all known choreographies, except the figure-eight, are linearly unstable [17, 18]. In [9], I show that the figure-eight orbit is linearly stable by using the strong amount of symmetry present to simplify the calculations. Specifically, I prove that the stability of the figure-eight can be determined from the first twelfth of its orbit. Using a clever change of coordinates, the problem is reduced to computing the eigenvalues of a 2×2 matrix whose entries depend on the solutions to the associated linear system. Although a Runge-Kutta-Fehlberg numerical algorithm is required to estimate these eigenvalues, only four steps of the algorithm are necessary to rigorously conclude linear stability. Although not a purely analytical proof, my reductions lead impressively to a short and straight-forward numerical calculation.

The reduction techniques developed for the figure-eight orbit should find other applications in the n-body problem as well as to the wider field of Hamiltonian dynamics. In particular, I look forward to applying these ideas to special symmetric periodic solutions in the n-body problem including other choreographies, solutions generated via variational methods with symmetry constraints (such as the recent solutions announced by Chen, et. al. [2]) and symmetric relative equilibria (such as the kite central configurations of the four-body problem). Based on my work, it appears that the only stable equal mass periodic solutions are those that exhibit a certain type of time-reversing symmetry.

In contrast to the figure-eight, the circular equilateral triangle solution of Lagrange is highly unstable in the equal mass case. In [11], I study the effects of adding eccentricity to the equilateral triangle solution. Given any relative equilibrium, it is possible to replace the circular orbit of each body with an elliptic solution of the Kepler problem, thereby creating an orbit that maintains the same shape, but changes its size. One of the interesting results I prove rigorously in [11] is that adding some eccentricity to a circular solution can change its stability type from unstable to stable. This provides some additional insight into the stability of the figure-eight. A variation in the size of the orbit may actually help it become more stable.

Another ingredient for stability, at least in the case of relative equilibria, is the existence of a dominant mass, a body significantly larger than all the others. The Lagrange equilateral triangle solution is linearly stable only when one mass dominates the other two. This is precisely the case in our solar system for the equilateral triangles formed by the sun, Jupiter and either the Trojan or Greek asteroids. Another example is the $(1 + n)$-gon relative equilibrium, formed by placing n equal masses at the vertices of a regular n-gon and adding an arbitrary mass at the center (a ring around a planet, for example). In [10], I show that this configuration is linearly stable if and only if the central mass is at least $0.435n^3$ larger than any mass on the ring. This evidence suggests that the only linearly stable relative equilibria contain a dominant mass, a conjecture often attributed to Rick Moeckel. The contrapositive to this conjecture is that *any* equal mass circular solution is unstable. In [13], I prove this is true although the estimates used require that there be at least $24,306$ bodies. Through the support of my current National Science Foundation (NSF) grant, I hope to improve upon this result, lowering this value to the expected number of two.

Recently, there has been a great deal of scholarly work in the field focused on *Saari's conjecture*, stating that the only solutions with constant moment of inertia (constant total size) are relative equilibria [16]. Since a rigid rotation doesn't alter the overall size of a solution, it is clear that any relative equilibrium has a constant moment of inertia. However, the converse is an open question that I have been investigating over the last few years.

In [12], I show that Saari's conjecture actually fails in certain Hamiltonian systems with power-law potential functions. This even includes the five-body problem if one allows "negative" masses. However, the conjecture has been verified by Moeckel for the three-body problem, regardless of the masses (as long as they are nonzero and their sum is nonzero) [7]. Moeckel's proof uses computational algebra, employing Bernstein-Khovanskii-Kushnirenko (BKK) theory to the problem. Following Moeckel's techniques, Lisa Melanson (HC '06) and I prove a version of Saari's conjecture for the planar, circular, restricted three-body problem (PCR3BP) in [15]. We show, without the use of a computer, that any solution to the PCR3BP with a constant amended potential (equivalently, a constant speed) must be a fixed point. What makes BKK theory so appealing is that it provides a relatively straightforward algorithm for determining if a system of polynomial equations has a finite number of solutions for which all variables are nonzero. Since Saari's conjecture can be formulated in terms of mutual distances that physically speaking, never vanish, BKK theory is ideally suited to address this challenging problem.

A major focus of my current NSF grant is to apply tools from algebraic geometry, such as BKK theory and Gröbner bases, to challenging questions in celestial mechanics. Building off the work Lisa and I began, Holy Cross seniors Julianne Kulevich and Christopher J. Smith have begun investigating Saari's conjecture for the restricted four-body problem using the software Maple and Matlab. The ultimate goal is to reduce the problem down to a small system of polynomial equations and then apply BKK theory to conclude that the number of solutions is finite. This is a significantly harder task due to the additional mass, leading to polynomials with thousands of terms. Hopefully, a positive result will prove relevant to the currently open Saari's conjecture for the full four-body problem.

Numerical Methods and Complex Dynamics

One of the most common iterative algorithms for finding solutions to an equation is *Newton's method*. Given an equation $f(x) = 0$ and an initial guess x_0, Newton's method provides a better guess given by

$$x_1 = N_f(x_0) = x_0 - \frac{f(x_0)}{f'(x_0)}.$$

Here, the numerical technique uses information about the first derivative of f at x_0 to obtain an improved approximation x_1. The process repeats, generating a sequence of numbers $x_0, x_1, x_2, \ldots$ that hopefully converges to a solution of the equation. More accurate numerical methods, such as *Halley's method*, utilize information from higher-order derivatives of f.

Interestingly, and perhaps of little surprise to computer users, numerical methods can fail quite drastically. For example, applying Newton's method to the equation $x^2 + 1 = 0$ with a real initial guess x_0 will *never* converge to a solution because the solutions are complex. Even worse, applying Newton's method to find the roots of $p(z) = z^4 - 6z^2 - 11$ leads to an *entire region* of the complex plane for which initial seeds eventually bounce back and forth between 1 and -1, neither of which are solutions to $p(z) = 0$. In this case, because $z = 1$ and $z = -1$ are critical points of N_p that attract nearby seeds, they are said to lie on a *superattracting* cycle of period two. It is this failure of the numerical method to converge on an open set of initial guesses that I find appealing.

I have studied this problem from a complex dynamical systems perspective. Given a complex polynomial $p(z)$, its roots are superattracting fixed points of the map determined by applying the numerical method to p. For the method to break down on an open set, there must be another attracting periodic cycle that draws in entire regions of the complex plane. An important theorem from complex dynamics, due to Julia and Fatou in the 1920's, states that the basin of an attracting cycle must contain a critical point of the map [1]. Therefore, by following the orbits of the "free" critical points, those other than the roots themselves, we can determine if an extraneous attracting cycle exists. The free critical points for Newton's method are the inflection points of p.

One useful approach I have employed is to study a given numerical method applied to a particular family of polynomials. In [14], Jeremy Horgan-Kobelski and I contrast Newton's and Halley's method applied to the family of cubics

$$p_\lambda(z) = (z-1)(z+1)(z-\lambda)$$

treating $\lambda \in \mathbb{C}$ as a complex parameter. By an affine conjugacy, this is equivalent to studying the problem for all cubics. Symmetry plays an important role here and we use it to explore the crucial case $\mathrm{Re}(\lambda) = 0$ where the roots form an isosceles triangle in the complex plane. The problem can be reduced to studying the dynamics of a rational map of a single real variable. Using this reduction, we show that there is a sequence of purely imaginary λ-values for which the corresponding numerical method applied to p_λ has a superattracting periodic cycle. Each value lies at the center of a Mandelbrot-like set in the complex parameter plane.

Trevor O'Brien (HC '05) and I tackled a similar, but more challenging problem, applying Newton's method to the quartic family

$$q_\lambda(z) = (z-1)(z+1)(z-\lambda)(z-\bar{\lambda}),$$

where $\lambda \in \mathbb{C}$ is a complex parameter. This family also has important symmetry but contains two free critical points to consider rather than one as in the cubic case. This problem turns out to have many similarities to Milnor's work on iterating the general complex cubic [6], in part because of the number of free critical points. Of particular interest is the discovery of figures in the parameter plane reminiscent of both quadratic phenomena (Mandelbrot-like sets and tricorns) as well as cubic behavior (swallow and product configurations). This discrepancy occurs because q''_λ has real coefficients and therefore the two free critical points are either real or complex conjugates. In the real case, there are two critical points to follow whose orbits are unrelated. In contrast, the orbits of a pair of complex conjugate critical points will be conjugate, so there is essentially only one critical point to examine, leading to quadratic-like behavior.

It is possible to explain much of the complicated parameter plane picture by following the orbits of the critical points of a real one-dimensional map. The large amount of symmetry present in our family leads to this nice reduction. Various phenomena in the parameter plane can be predicted. Using a bisection algorithm, we numerically locate an abundance of λ-values along the imaginary axis for which Newton's method applied to q has a superattracting periodic cycle. The period and type of attracting cycle located governs much of the structure in the parameter plane along the imaginary axis. It should be possible to use techniques from real dynamical systems theory to rigorously prove more of our discoveries. I look forward to continuing the work on this enticing problem.

Accomplishments

To date, I have written or co-authored twelve refereed papers, eleven of which have appeared in or been accepted to respected international journals. Two of these publications were co-authored with undergraduates. I have delivered a multitude of talks and am routinely invited to speak at major conferences in celestial mechanics. This past May, I was one of five invited speakers to present a mini-course on my research at the National Center for Theoretic Sciences (NCTS) Workshop on Dynamical Systems, hosted by NCTS in Hsinchu, Taiwan. I have given invited talks in eight different countries and seventeen different states. In January 2004, I co-organized a "Special Session on Celestial Mechanics" at the Joint AMS/MAA Mathematics Meetings in Phoenix, Arizona. I have also served as a referee for papers in eleven different journals and have written seventeen published reviews for *Mathematical Reviews*.

Research Overview

Since coming to Holy Cross I have been fortunate to receive excellent financial support for scholarship activities from the college as well as outside funding agencies. This includes two Holy Cross Batchelor Ford Foundation Grants in the summers of 2002 and 2007, a Council on Undergraduate Research Student Summer Research Fellowship in 2004 and most recently, a three-year award from the National Science Foundation (NSF DMS-0708741) beginning in July 2007. All four of my grant proposals specified research with undergraduates as a significant component. Indeed, I have been delighted to find several outstanding undergraduate researchers at Holy Cross and have chosen to make collaboration with undergraduates a major aspect of my scholarship.

Currently, I am working with Holy Cross seniors Julianne Kulevich and Christopher J. Smith on some interesting questions in the restricted four-body problem. Our research grew out of work I conducted with Lisa Melanson (HC '06) during the summer and fall of 2005. Lisa and I successfully applied BKK Theory to prove a result in the restricted three-body problem that was published in the journal *Celestial Mechanics and Dynamical Astronomy* [15]. I have also mentored Holy Cross students Trevor O'Brien (HC '05) and Gabe Weaver (HC '04). Trevor completed a departmental honors thesis under my direction entitled "Elusive Zeros Under Newton's Method." Based on our work, we have been invited to submit an article for an upcoming book on chaos and fractals to be published by the Mathematical Association of America. While an NSF Vigre Postdoc at the University of Colorado, Boulder, I worked with undergraduate Jeremy Horgan-Kobelski on comparing Newton's and Halley's method for cubic polynomials. Our collaborative efforts led to a joint paper in the *International Journal of Bifurcation and Chaos*, featuring one of our figures on the cover of the journal [14].

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