> Some symbolic calculations used in the paper "Four-body co-circular central configurations" by Josep M. Cors and Gareth E. Roberts, to appear in *Nonlinearity*.

>

> $factor\left(r_{23}^2 \cdot r_{24}^2 \cdot \left(r_{13}^3 - r_{14}^3\right) - r_{13}^2 \cdot r_{14}^2 \cdot \left(r_{24}^3 - r_{23}^3\right) - r_{13}^2 \cdot r_{23}^2 \cdot \left(r_{24}^3 - r_{14}^3\right) + r_{14}^2 \cdot r_{24}^2 \cdot \left(r_{13}^3 - r_{23}^3\right)\right)$

$$\left(r_{13} - r_{24}\right)\left(r_{13}^2 r_{23}^2 r_{24}^2 + r_{13}^2 r_{14}^2 r_{24}^2 + r_{13} r_{23}^2 r_{14}^3 + r_{13} r_{14}^2 r_{23}^3 + r_{24} r_{23}^2 r_{14}^3 + r_{24} r_{14}^2 r_{23}^3\right) \quad (1)$$

> This confirms the factorization given in the proof of Lemma 3.2.

Trapezoid: Rigorous Bound, y > 9/10

> $with(Groebner):$

> $T := \left(y^2 + x\right)^{\frac{3}{2}} \cdot \left(2 \cdot y^3 - x^3 - 1\right) - y^3 - x^3 \cdot y^3 + 2 \cdot x^3:$

> $eq1 := z^3 \cdot \left(2 \cdot y^3 - x^3 - 1\right) - y^3 - x^3 \cdot y^3 + 2 \cdot x^3:$

> $eq2 := x + y^2 - z^2:$

> $bd1 := simplify\left(1000 \cdot subs\left(y = \frac{9}{10}, eq1\right)\right):$

> $bd2 := simplify\left(100 \cdot subs\left(y = \frac{9}{10}, eq2\right)\right):$

> $NList := [bd1, bd2]:$

>

> $Bd := Basis(NList, plex(x, z)):$

> $factor(Bd[1])$

$$1404461511 - 989441000 z^3 + 1968300000 z^5 - 2501709300 z^2 + 3088530000 z^4 - 1271000000 z^6 - 2430000000 z^7 + 1000000000 z^9 \quad (1.1)$$

> $fsolve(Bd[1])$

$$-1.266884139 \quad (1.2)$$

> $s := sturmseq(Bd[1], z):$

> $sturm(s, z, 0, 2);$

$$0 \quad (1.3)$$

> This shows that the level curve T = 0 does not intersect y = 9/10 in Lambda. Consequently, y > 9/10 is required by uniqueness and continuity of T=0.

>

>

Finding the minimum on the trapezoid curve τ(x)

> $with(Groebner):$

> $T := (y^2+x)^{\frac{3}{2}} \cdot (2 \cdot y^3 - x^3 - 1) - y^3 - x^3 \cdot y^3 + 2 \cdot x^3:$

> $Tx := (y^2+x)^{\frac{1}{2}} \cdot (2 \cdot y^3 - 3 \cdot x^3 - 2 \cdot x^2 \cdot y^2 - 1) - 2 \cdot x^2 \cdot (y^3 - 2):$

> $t1 := z^3 \cdot (2 \cdot y^3 - x^3 - 1) - y^3 - x^3 \cdot y^3 + 2 \cdot x^3:$

> $t2 := z \cdot (2 \cdot y^3 - 3 \cdot x^3 - 2 \cdot x^2 \cdot y^2 - 1) - 2 \cdot x^2 \cdot (y^3 - 2):$

> $t3 := x + y^2 - z^2:$

>

> $f := expand\left((x+y^2)^3 \cdot (2 \cdot y^3 - x^3 - 1)^2 - (y^3 + x^3 \cdot y^3 - 2 \cdot x^3)^2\right):$

> $g := expand\left((x+y^2) \cdot (2 \cdot y^3 - 3 \cdot x^3 - 2 \cdot x^2 \cdot y^2 - 1)^2 - 4 \cdot x^4 \cdot (y^3 - 2)^2\right):$

>

> $PList := [t1, t2, t3]:$

> $B1 := Basis(PList, plex(z, x, y)):$

> $tr1 := simplify\left(\frac{B1[1]}{y^3}\right):$

> $tr1 - res2$

$$0 \qquad (2.1)$$

> $fsolve(tr1)$

$$-0.8930736485, 0.9080259298, 1.344741734 \qquad (2.2)$$

> $B2 := Basis(PList, plex(z, y, x)):$

> $tr2 := simplify\left(\frac{B2[1]}{x^2 \cdot (x-1) \cdot (x^2+x+1)}\right):$

> $fsolve(tr2)$

$$-1.716120172, -0.4974351142, 0.6035381491 \qquad (2.3)$$

>

> $s1 := sturmseq(tr1, y): \ s2 := sturmseq(tr2, x):$

> $sturm(s1, y, 0, 1);$

$$1 \qquad (2.4)$$

> $sturm(s2, x, 0, 1);$

$$1 \qquad (2.5)$$

> **This shows that there is precisely one physical solution to tau'(x) = 0 given by x = 0.6035381491 and y = 0.9080259298.**

>

> **Below are the calculations obtained by squaring both sides of T = 0 and T_x = 0.**

> $PList := [f, g]:$

> $Blex := Basis(PList, plex(x, y)):$

> $factor(Blex[1]):$

> $res := resultant(f, g, x):$

> $factor(res):$

> $res1 := 15872\,y^{45} - 247353\,y^{42} + 1799812\,y^{39} - 8099516\,y^{36} + 25109720\,y^{33} - 56519861\,y^{30} + 94946448\,y^{27} - 120711684\,y^{24} + 116849376\,y^{21} - 86015143\,y^{18} + 47564748\,y^{15} - 19074048\,y^{12} + 5169384\,y^{9} - 853979\,y^{6} + 77824\,y^{3} - 4096:$

> $fsolve(res1)$

$$0.9292859667 \quad (2.6)$$

> $res2 := 16384\,y^{39} - 194263\,y^{36} + 958540\,y^{33} - 2539172\,y^{30} + 3846072\,y^{27} - 3109243\,y^{24} + 622144\,y^{21} + 1205860\,y^{18} - 1230512\,y^{15} + 518263\,y^{12} - 102604\,y^{9} + 7504\,y^{6} + 216\,y^{3} + 27:$

> $fsolve(res2)$

$$-0.8930736485,\ 0.9080259298,\ 1.344741734 \quad (2.7)$$

> $fy := subs(y = 0.9080259298, g):$

> $fsolve(fy);$

$$-0.3430330869,\ 0.3172090634,\ 0.6035381491 \quad (2.8)$$

> $fytest := subs(y = 0.9292859667, g):$

> $fsolve(fytest);$

$$-0.3871957355,\ 0.3495940365,\ 0.6146710034 \quad (2.9)$$

> **Checking all the found solutions in the region shows that only x = .6035381491, y = .9080259298 satisfies both T = 0 and T_x = 0.**

>

> $subs(\{x = 0.6035381491, y = 0.9080259298\}, Tx);$

$$1.1\ 10^{-9} \quad (2.10)$$

> $subs(\{x = 0.3172090634, y = 0.9080259298\}, Tx);$

$$0.5036402591 \quad (2.11)$$

> $subs(\{x = 0.3495940365, y = 0.9292859667\}, Tx);$

$$0.5854117993 \quad (2.12)$$

> $subs(\{x = 0.6146710034, y = 0.9292859667\}, T);$

$$0.1455827582 \quad (2.13)$$

>

Trapezoid: Mass m is an increasing function of x

> $with(Groebner):$

> $T := (y^2 + x)^{\frac{3}{2}} \cdot (2 \cdot y^3 - x^3 - 1) - y^3 - x^3 \cdot y^3 + 2 \cdot x^3:$

> $Tx := \left(\frac{3}{2}\right) \cdot (y^2 + x)^{\frac{1}{2}} \cdot (2 \cdot y^3 - 3 \cdot x^3 - 2 \cdot x^2 \cdot y^2 - 1) - 3 \cdot x^2 \cdot (y^3 - 2):$

> $Tx1 := \left(\frac{3}{2}\right)\cdot z\cdot(2\cdot y^3 - 3\cdot x^3 - 2\cdot x^2\cdot y^2 - 1) - 3\cdot x^2\cdot(y^3 - 2):$

> $Ty := 3\cdot y\cdot\left((y^2+x)^{\frac{1}{2}}\cdot(4\cdot y^3 - x^3 - 1 + 2\cdot x\cdot y) - y\cdot(x^3+1)\right):$

> $Ty1 := 3\cdot y\cdot(z\cdot(4\cdot y^3 - x^3 - 1 + 2\cdot x\cdot y) - y\cdot(x^3+1)):$

> $eq1 := z^3\cdot(2\cdot y^3 - x^3 - 1) - y^3 - x^3\cdot y^3 + 2\cdot x^3:$

> $eq2 := (1 - y^3)\cdot(2\cdot y^3 + x^3)\cdot Ty1 + 3\cdot x\cdot y^2\cdot(1 - x^3)\cdot Tx1:$

> $eq3 := x + y^2 - z^2:$

> $PList := [eq1, eq2, eq3]:$

> $B := Basis(PList, plex(z, y, x)):$

>

> $msder := simplify\left(\frac{B[1]}{x^4\cdot(x-1)^2\cdot(x^2+x+1)^2}\right);$

$$msder := 12252303 - 388072944\,x^3 + 12252303\,x^{48} + 11679662896\,x^9 + 2198862536\,x^6 + 34333498404\,x^{12} + 709488857616\,x^{15} + 1980505257336\,x^{18} + 3753406853296\,x^{21} + 4259444325914\,x^{24} + 3753406853296\,x^{27} + 1980505257336\,x^{30} + 709488857616\,x^{33} + 34333498404\,x^{36} + 11679662896\,x^{39} + 2198862536\,x^{42} - 388072944\,x^{45} \quad (3.1)$$

> $fsolve(msder);$

$$-1.709770733,\ -0.5848737381 \quad (3.2)$$

> $s := sturmseq(msder, x):$

> $sturm(s, x, 0, 1);$

$$0 \quad (3.3)$$

> **This shows that there is no intersection between T(x,y) = 0 and the numerator of the partial derivative of m with respect to x. Consequently, m'(x) > 0 except at x=0 and x = 1.**

Trapezoid: Circumradius is an increasing function of x

> $with(Groebner):$

> $T := (y^2 + x)^{\frac{3}{2}}\cdot(2\cdot y^3 - x^3 - 1) - y^3 - x^3\cdot y^3 + 2\cdot x^3:$

> $Tx := \left(\frac{3}{2}\right)\cdot(y^2 + x)^{\frac{1}{2}}\cdot(2\cdot y^3 - 3\cdot x^3 - 2\cdot x^2\cdot y^2 - 1) - 3\cdot x^2\cdot(y^3 - 2):$

$> Tx1 := \left(\frac{3}{2}\right)\cdot z\cdot(2\cdot y^3 - 3\cdot x^3 - 2\cdot x^2\cdot y^2 - 1) - 3\cdot x^2\cdot(y^3 - 2):$

$> Ty := 3\cdot y\cdot\left((y^2+x)^{\frac{1}{2}}\cdot(4\cdot y^3 - x^3 - 1 + 2\cdot x\cdot y) - y\cdot(x^3+1)\right):$

$> Ty1 := 3\cdot y\cdot(z\cdot(4\cdot y^3 - x^3 - 1 + 2\cdot x\cdot y) - y\cdot(x^3+1)):$

>

$> eq1 := z^3\cdot(2\cdot y^3 - x^3 - 1) - y^3 - x^3\cdot y^3 + 2\cdot x^3:$

$> eq2 := y\cdot(2\cdot y^2 + 2\cdot x\cdot y^2 + x^2 - 1)\cdot Ty1 - (8\cdot y^4 - 2\cdot(1-x)^2\cdot(x+2\cdot y^2))\cdot Tx1:$

$> eq3 := x + y^2 - z^2:$

> PList := [eq1, eq2, eq3]:

> B := Basis(PList, plex(z, x, y)):

> factor(B[1]):

$> rcder := simplify\left(\frac{B[1]}{y^4\cdot(y-1)\cdot(y^2+y+1)}\right);$

$$rcder := 301754331\,y^{56} + 1296 + 91539\,y^4 - 4867806564\,y^{47} + 10733904\,y^{60} + 8305068\,y^{13} - 2806524093\,y^{24} - 1278\,y^7 - 11796\,y^2 + 12636\,y^3 + 4240796649\,y^{50} + 7010912685\,y^{30} + 3409455612\,y^{25} + 28398879276\,y^{35} + 1620876\,y^{11} - 251552\,y^6 - 2293131\,y^{10} + 296196\,y^8 - 89098\,y^9 + 8660727447\,y^{26} - 5272605084\,y^{49} + 7675033045\,y^{36} - 9244521774\,y^{29} - 513419601\,y^{54} - 73431247\,y^{18} - 580401\,y^{16} + 27296461534\,y^{45} - 32085603117\,y^{42} + 578922216\,y^{52} - 36720479115\,y^{38} - 77196\,y^5 - 5574620427\,y^{32} + 3853302060\,y^{51} - 18716289\,y^{14} - 4003797081\,y^{44} + 1984308056\,y^{27} - 13436554879\,y^{48} - 834624\,y^{61} + 18144\,y^{62} - 16151978114\,y^{33} + 52009534917\,y^{40} - 3146087532\,y^{23} + 16408243528\,y^{39} + 23062367382\,y^{41} + 19016171211\,y^{46} + 422672709\,y^{20} - 17008488\,y^{58} + 11813703\,y^{12} - 19798812\,y^{57} - 45302568\,y^{15} + 928142026\,y^{21} - 1579155150\,y^{53} + 74343228\,y^{55} - 24577128\,y^{59} - 11203627599\,y^{28} + 12694871178\,y^{31} - 209832495\,y^{22} + 6870658455\,y^{34} - 37831461708\,y^{37} + 36978546\,y^{17} - 40072132422\,y^{43} - 44600322\,y^{19} \qquad \textbf{(4.1)}$$

> sort(rcder, y);

$$18144\,y^{62} - 834624\,y^{61} + 10733904\,y^{60} - 24577128\,y^{59} - 17008488\,y^{58} - 19798812\,y^{57} + 301754331\,y^{56} + 74343228\,y^{55} - 513419601\,y^{54} - 1579155150\,y^{53} + 578922216\,y^{52} + 3853302060\,y^{51} + 4240796649\,y^{50} - 5272605084\,y^{49} - 13436554879\,y^{48} - 4867806564\,y^{47} + 19016171211\,y^{46} + 27296461534\,y^{45} - 4003797081\,y^{44} \qquad \textbf{(4.2)}$$

$$- 40072132422\, y^{43} - 32085603117\, y^{42} + 23062367382\, y^{41} + 52009534917\, y^{40} + 16408243528\, y^{39} - 36720479115\, y^{38} - 37831461708\, y^{37} + 7675033045\, y^{36} + 28398879276\, y^{35} + 6870658455\, y^{34} - 16151978114\, y^{33} - 5574620427\, y^{32} + 12694871178\, y^{31} + 7010912685\, y^{30} - 9244521774\, y^{29} - 11203627599\, y^{28} + 1984308056\, y^{27} + 8660727447\, y^{26} + 3409455612\, y^{25} - 2806524093\, y^{24} - 3146087532\, y^{23} - 209832495\, y^{22} + 928142026\, y^{21} + 422672709\, y^{20} - 44600322\, y^{19} - 73431247\, y^{18} + 36978546\, y^{17} - 580401\, y^{16} - 45302568\, y^{15} - 18716289\, y^{14} + 8305068\, y^{13} + 11813703\, y^{12} + 1620876\, y^{11} - 2293131\, y^{10} - 89098\, y^{9} + 296196\, y^{8} - 1278\, y^{7} - 251552\, y^{6} - 77196\, y^{5} + 91539\, y^{4} + 12636\, y^{3} - 11796\, y^{2} + 1296$$

> $fsolve(rcder);$

-0.9192613441, -0.8835259921, -0.8806859995, -0.7897153641, 0.6309394391, 1.501181561, 20.91822519, 21.93871288 **(4.3)**

> $s := sturmseq(rcder, y):$

> $sturm\left(s, y, \frac{9}{10}, 1\right);$

$$0 \quad \textbf{(4.4)}$$

> $subs(\{x = 0, y = 1, z = 1\}, PList);$

$$[0, 0, 0] \quad \textbf{(4.5)}$$

> $subs(\{x = 1, y = 1, z = \mathrm{sqrt}(2)\}, PList);$

$$\left[0, 96\sqrt{2} - 48, 0\right] \quad \textbf{(4.6)}$$

> **This shows that there is no intersection between T(x,y) = 0 and the numerator of**
> **the partial derivative of r_c^2 with respect to x.**
> **Consequently, r_c'(x) > 0 except at**
> **x=0.**

>

> **Note: There does happen to be a solution to the above system but with a negatize z value, so it can be ignored.**

> $\mathrm{sqrt}(0.2038308484 + 0.6309394391^2);$

$$0.7758320851 \quad \textbf{(4.7)}$$

> $subs(\{x = 0.2038308484, y = 0.6309394391, z = -0.7758320851\}, PList);$

$$[1.\,10^{-11}, -0., -1.\,10^{-10}] \quad \textbf{(4.8)}$$

>

Trapezoid: θ14 is a decreasing function of x

```
> with(Groebner):
> T := (y^2 + x)^(3/2)·(2·y^3 - x^3 - 1) - y^3 - x^3·y^3 + 2·x^3:
> Tx := (3/2)·(y^2 + x)^(1/2)·(2·y^3 - 3·x^3 - 2·x^2·y^2 - 1) - 3·x^2·(y^3 - 2):
> Tx1 := (3/2)·z·(2·y^3 - 3·x^3 - 2·x^2·y^2 - 1) - 3·x^2·(y^3 - 2):
> Ty := 3·y·((y^2 + x)^(1/2)·(4·y^3 - x^3 - 1 + 2·x·y) - y·(x^3 + 1)):
> Ty1 := 3·y·(z·(4·y^3 - x^3 - 1 + 2·x·y) - y·(x^3 + 1)):
>
> eq1 := z^3·(2·y^3 - x^3 - 1) - y^3 - x^3·y^3 + 2·x^3:
> eq2 := (-2·y^2 - 2·x·y^2 - x^2 + 1)·Ty1 - 2·y(x + 1)^2·Tx1:
> eq3 := x + y^2 - z^2:
> PList := [eq1, eq2, eq3]:
> B := Basis(PList, plex(z, y, x)):
> factor(B[1]):
> r := simplify(B[1] / (x^2·(x + 1)·(x^2 + x + 1)^2·(x - 1)^3)):
>
> sort(r, x);
```

$$63x^{42} - 768x^{41} + 2424x^{40} + 7166x^{39} - 48840x^{38} + 12768x^{37} + 395941x^{36} - 211872x^{35} - 2405712x^{34} - 751028x^{33} + 12953520x^{32} + 27357408x^{31} + 14860695x^{30} + 82331232x^{29} + 961338312x^{28} + 4780976354x^{27} + 15280798344x^{26} + 36424232640x^{25} + 68944782213x^{24} + 107067631296x^{23} + 138590661792x^{22} + 150912729512x^{21} + 138590661792x^{20} + 107067631296x^{19} + 68944782213x^{18} + 36424232640x^{17} + 15280798344x^{16} + 4780976354x^{15} + 961338312x^{14} + 82331232x^{13} + 14860695x^{12} + 27357408x^{11} + 12953520x^{10} - 751028x^{9} - 2405712x^{8} - 211872x^{7} + 395941x^{6} + 12768x^{5} - 48840x^{4} + 7166x^{3} + 2424x^{2} - 768x + 63 \quad \textbf{(5.1)}$$

```
> fsolve(r);
```

$$-1.954190225, -0.5117209099 \quad \textbf{(5.2)}$$

```
> s := sturmseq(r, x):
> sturm(s, x, 0, 1);
```

$$0 \quad \textbf{(5.3)}$$

```
>
> subs({x = 1, y = 1, z = sqrt(2)}, PList);
```

(5.4)

$$[0, 0, 0] \quad (5.4)$$

```
> This shows that there is no intersection between T(x,y) = 0 and
  the numerator of
  the partial derivative of (y^2/r_c^2) with respect to x.
  Consequently, d(y/r_c)/dx < 0 except at x=1 (square) where it
  equals 0.
>
```