Math/Music: Aesthetic Links

Homework Assignment #4 (Last One!)

DUE DATE: Tuesday, April 3, 4:00 pm

Homework should be turned in either directly to me or placed in the black folder outside my office door. You should write up solutions neatly to all problems, making sure to show all your work. You are encouraged to work on these problems with other classmates, although the solutions you turn in should be your own work. Please cite any references (web based or text) that you may have used for assistance with the assignment.

Note: Please list the names of any students or faculty who you worked with on the assignment.

> *[I]t should be frankly admitted that in some plants the numbers do not belong to the sequence of f's [Fibonacci numbers] but to the sequence of g's [Lucas numbers] or even to the still more anomalous sequences*
>
> $$3, 1, 4, 5, 9, \ldots \quad \textit{or} \quad 5, 2, 7, 9, 16, \ldots$$
>
> *Thus we must face the fact that [Fibonacci] phyllotaxis is really not a universal law but only a fascinatingly prevalent tendency.*
>
> –H. S. M. Coxeter, from *Introduction to Geometry* (1961, Wiley, p. 172)

1. **Fibonacci in Nature:** Find a flower somewhere on campus and count the number of petals. Do you find a Fibonacci number? Given the current season, you may have to be creative where you look to locate a flower. State the location, type of flower and number of petals you counted.

2. **Fibonacci in Music:** Suppose you have a piece of music which is 144 measures long. Where would you put the climax of the piece if you wanted to divide it into two parts whose ratio was the golden ratio? There are two answers – find both (round to the nearest measure). What is significant about these numbers? Do the same problem for a piece of music which is 150 measures long. Why do you have to round off by more to get your answer with 150 than with 144? *Hint:* This last question has something to do with properties of continued fractions.

3. **The Golden Ratio:** Enter the golden ratio ϕ into your calculator, square it, and note the result. Then enter ϕ again, take its reciprocal $1/\phi$, and compare this to the previous result. What do you observe? Construct an equation that reflects your observation, and then prove that your equation is correct.

4. **Fibonacci and the Golden Ratio:** In class we observed that

   $$\lim_{n \to \infty} \frac{F_{n+1}}{F_n} = \phi = \frac{1 + \sqrt{5}}{2}.$$

   In other words, the ratios of successive Fibonacci numbers get closer and closer to the golden ratio as you move further out in the Fibonacci sequence. This key fact is what relates the Fibonacci numbers to the golden ratio. In this exercise you will prove this important fact (given one key assumption).

a. Start with the relation $F_{n+1} = F_n + F_{n-1}$ and divide both sides by F_n. Simplify.

b. We will assume that $\lim_{n\to\infty} \dfrac{F_{n+1}}{F_n} = L$ exists. This requires proof, but we'll assume that the limit exists and call it L since we don't know yet that it is really ϕ. Given that this limit equals L, what is $\lim_{n\to\infty} \dfrac{F_n}{F_{n-1}}$?

c. Rewrite the right-hand side of your simplified equation from part **a.** to include the ratio F_n/F_{n-1}. Now take the limit as $n \to \infty$ of both sides to get an equation solely in terms of L. You can apply such facts as "the limit of a sum equals the sum of the limits."

d. Solve your equation for L from part **c.** and show that $L = \phi$. Why does this prove that regardless of what two initial seeds you begin with, the ratio of successive terms always approaches the golden ratio?!

5. **Fibonacci Identities:** Find formulas for the following Fibonacci identities. A typical answer will look something like $F_{n+1} + 2$. Show some evidence as to how you arrived at your answers.

   a. $F_1 + F_3 + \cdots + F_{2n-1} =$ ____________________.

   b. $F_1 + F_2 + \cdots + F_n =$ ____________________.

   c. $F_n F_{n+2} - F_{n+1}^2 =$ ____________________.

   Bonus Question: Using the fact that $F_2 = F_3 - F_1, F_3 = F_4 - F_2, \ldots$, prove your answer to part **b.** *Hint:* "I Can See for Miles and Miles"