Change Ringing

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Math/Music: Aesthetic Links
Montserrat Seminar Spring 2012
February 6 and 8, 2012

Change Ringing (Bell Ringing)

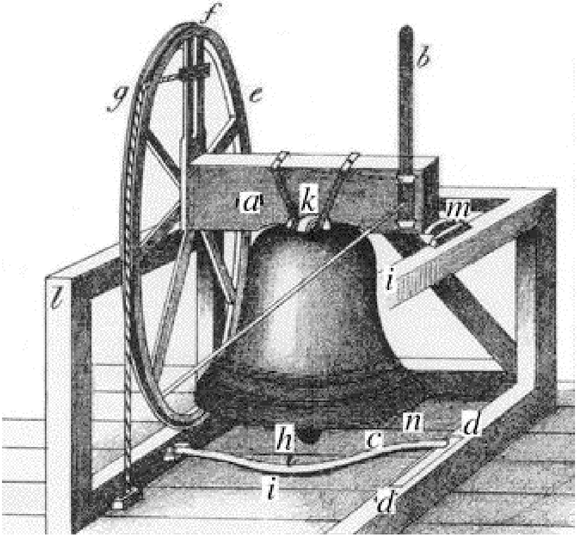

Figure: A typical church bell rung in the belfry.

Figure: Bell ringing practice in Stoke Gabriel parish church, south Devon, England.

Change ringing is a non-competitive and non-violent team activity that is highly stimulating intellectually and mildly demanding physically, and makes a beautiful sound. It develops mental and physical skills in a context of communal effort. The intense concentration required brings euphoric detachment that cleanses the mind of the day's petty demands and frustrations.

North American Guild of Change Ringers

Figure: The Swan Bells Tower in Perth, Australia, a unique icon for Western Australia. Contains 12 royal bells from England (St. Martin-in-the-Fields).

Figure: Bell ringing demonstration in Swan Bell Tower.

Change Ringing: An Example

1 2 3 4	1 3 4 2	1 4 2 3
2 1 4 3	3 1 2 4	4 1 3 2
2 4 1 3	3 2 1 4	4 3 1 2
2 4 3 1	3 2 4 1	4 3 2 1
4 2 3 1	2 3 4 1	3 4 2 1
4 2 1 3	2 3 1 4	3 4 1 2
4 1 2 3	2 1 3 4	3 1 4 2
1 4 3 2	1 2 4 3	1 3 2 4
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Canterbury Minimus (true extent on 4 bells)

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4 2 3 1	2 3 4 1	3 4 2 1
4 2 1 3	2 3 1 4	3 4 1 2
4 1 2 3	2 1 3 4	3 1 4 2
1 4 3 2	1 2 4 3	1 3 2 4
		1 2 3 4

Canterbury Minimus (true extent on 4 bells)

There are $4! = 24$ different possible rows. Each must be rung exactly once starting and ending with rounds (1 2 3 4).

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Rules to ring an extent on n bells:

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Note: Rules 1 - 3 are mandatory for an extent while Rules 4 - 6 are optional though often satisfied.

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HW #2: List the moves for $n = 5$ and $n = 6$ bells. Find the pattern in the sequence of the number of allowable moves.

n	$n!$	Approximate Duration	Name
3	6	15 secs.	*Singles*
4	24	1 mins.	*Minimus*
5	120	5 mins.	*Doubles*
6	720	30 mins.	*Minor*
7	5,040	3 hrs.	*Triples*
8	40,320	24 hrs.	*Major*
9	362,880	9 days	*Caters*
10	3,628,800	3 months	*Royal*
11	39,916,800	3 years	*Cinques*
12	479,001,600	36 years	*Maximus*

Table: Approximate duration to ring an extent on n bells and the names given to such an extent. Compositions: *Plain Bob Minimus*, *Grandshire Triples*

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1 2 3	1 2 3
2 1 3	1 3 2
2 3 1	3 1 2
3 2 1	3 2 1
3 1 2	2 3 1
1 3 2	2 1 3
1 2 3	1 2 3

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1 2 3	1 2 3
2 1 3	1 3 2
2 3 1	3 1 2
3 2 1	3 2 1
3 1 2	2 3 1
1 3 2	2 1 3
1 2 3	1 2 3

Note the simple zig-zag pattern of Bell 1 in the first extent, sweeping easily from position 1 to position 3 and back again. We say that Bell 1 is *plain hunting.* It only needs to do this once to complete the extent. In this case, we say that the bell is "not working." Notice that in the second extent, Bell 1 follows a similar zig-zag path except that this begins on the second change.

Change Ringing

Plain Bob Minimus (read down first, then hop to next column)

1 2 3 4	1 3 4 2	1 4 2 3
2 1 4 3	3 1 2 4	4 1 3 2
2 4 1 3	3 2 1 4	4 3 1 2
4 2 3 1	2 3 4 1	3 4 2 1
4 3 2 1	2 4 3 1	3 2 4 1
3 4 1 2	4 2 1 3	2 3 1 4
3 1 4 2	4 1 2 3	2 1 3 4
1 3 2 4	1 4 3 2	1 2 4 3
		1 2 3 4

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2 4 1 3	3 2 1 4	4 3 1 2
4 2 3 1	2 3 4 1	3 4 2 1
4 3 2 1	2 4 3 1	3 2 4 1
3 4 1 2	4 2 1 3	2 3 1 4
3 1 4 2	4 1 2 3	2 1 3 4
1 3 2 4	1 4 3 2	1 2 4 3
		1 2 3 4

Let $a = (12)(34), b = (23), c = (34)$. The above sequence of 24 permutations can be "factored" as

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4 2 3 1	2 3 4 1	3 4 2 1
4 3 2 1	2 4 3 1	3 2 4 1
3 4 1 2	4 2 1 3	2 3 1 4
3 1 4 2	4 1 2 3	2 1 3 4
1 3 2 4	1 4 3 2	1 2 4 3
		1 2 3 4

Let $a = (12)(34), b = (23), c = (34)$. The above sequence of 24 permutations can be "factored" as

$$[(ab)^3 ac]^3 \; = \; [abababa\, c]^3 \qquad \text{Palindrome!}$$