# Mozart's Piano Sonatas and the Golden Ratio

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## Wolfang Amadeus Mozart, 1756-1791

- Musical prodigy – plays piano well by age 4, composes short piano pieces at age 5, talented violinist. Travels around Europe showing off his talents for European royalty.
- Interested in arithmetic at an early age. Scribbled figures and calculations all over the walls of his house and that of the neighbors.
- At 14, writes to his sister Nannerl asking her to send him arithmetic tables and more problems in arithmetic for "fun."
- Margins of the manuscript for *Fantasia and Fugue in C major* contain calculations on the probability of winning a lottery.
- "The pleasure of playing with figures remained with Mozart all his life long. Thus he once took up the problem, very popular at the time, of composing minuets 'mechanically,' by putting two-measure melodic fragments together in any order." (Alfred Einstein, biographer of Mozart)

# Mozart and Form

- Mozart known for his wonderful, charismatic melodies.
- Also well known for his structure and form – balance.
- In 1853, Henri Amiel wrote, “the balance of the whole [in Mozart’s music] is perfect.”
- Hanns Dennerlein characterized Mozart’s music as containing “the most exalted proportions,” and that Mozart possessed “an inborn sense for proportions.”
- According to Eric Blom, Mozart had “an infallible taste for saying exactly the right thing at the right time and at the right length.”

# Piano Sonatas

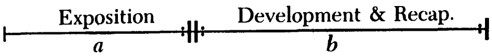


**FIGURE 3**
Sonata-form movement.

- Begins composing piano sonatas at the age of 18.
- Writes a total of 19 piano sonatas, almost all of them with three movements, each in sonata form.
- Sonata form consists of two main parts: An Exposition in which the main musical theme is announced and the Development and Recapitulation in which the theme is developed, often expanded and then revisited to finish the movement.
- Both the Exposition and the Development and Recapitulation were repeated to help demarcate the form to the listener.

TABLE 1

| Köchel | $a$ | $b$ | $a+b$ |
|---|---|---|---|
| 279, I | 38 | 62 | 100 |
| 279, II | 28 | 46 | 74 |
| 279, III | 56 | 102 | 158 |
| 280, I | 56 | 88 | 144 |
| 280, II | 24 | 36 | 60 |
| 280, III | 77 | 113 | 190 |
| 281, I | 40 | 69 | 109 |
| 281, II | 46 | 60 | 106 |
| 282, I | 15 | 18 | 33 |
| 282, III | 39 | 63 | 102 |
| 283, I | 53 | 67 | 120 |
| 283, II | 14 | 23 | 37 |
| 283, III | 102 | 171 | 273 |
| 284, I | 51 | 76 | 127 |
| 309, I | 58 | 97 | 155 |
| 311, I | 39 | 73 | 112 |
| 310, I | 49 | 84 | 133 |
| 330, I | 58 | 92 | 150 |
| 330, III | 68 | 103 | 171 |
| 332, I | 93 | 136 | 229 |
| 332, III | 90 | 155 | 245 |
| 333, I | 63 | 102 | 165 |
| 333, II | 31 | 50 | 81 |
| 457, I | 74 | 93 | 167 |
| 533, I | 102 | 137 | 239 |
| 533, II | 46 | 76 | 122 |
| 545, I | 28 | 45 | 73 |
| 547a, I | 78 | 118 | 196 |
| 570, I | 79 | 130 | 209 |

Figure: Mozart's piano sonatas and their division into two sections based on sonata form. From *The Golden Section and the Piano Sonatas of Mozart* by John Putz, *Mathematics Magazine*, 1995.

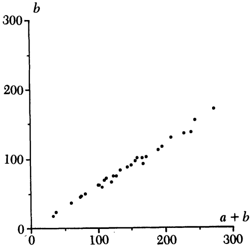


**FIGURE 4**

Scatter plot of $b$ against $a+b$.

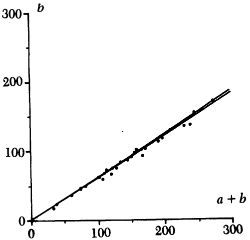


**FIGURE 5**

Scatter plot of $b$ and $a+b$ with the line $y=\varphi x$ (top) and the regression line (bottom).

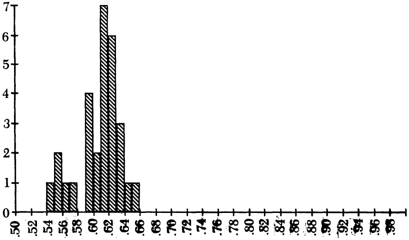


**FIGURE 6**

Frequency distribution of $\dfrac{b}{a+b}$.

## Analysis of the Data

- Data looks very convincing. For example, in the 1st sonata, dividing the total piece of 100 measures by the golden ratio gives 62 (rounded to the nearest integer), which is exactly the length Mozart uses.
- Similarly, the second movement of the 1st sonata has $74/\phi \approx 46$, again a near perfect division based on the golden ratio.
- However, the third movement should have a division at

$$158/\phi \;\approx\; 98 \;\neq\; 102.$$

- The $r^2$ value for the linear fit (a measure of how linear the data is) is $r^2 = 0.990$ ($r^2 = 1$ is a perfect fit). This shows a remarkably good approximation.

## On the Other Hand

- If the proportion of the largest to the whole $b/(a+b)$ is close to the golden ratio, then so should the proportion of the smallest to the largest $a/b$.

$$\frac{b}{a+b} \approx \frac{a}{b} \approx \frac{1}{\phi} = \phi - 1 \approx 0.6180$$

- What does the plot look like if we compare $a$ and $b$? Do we see an equally impressive correlation?

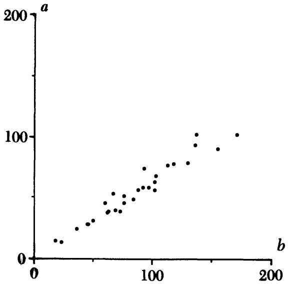


**FIGURE 7**

Scatter plot of $a$ against $b$.

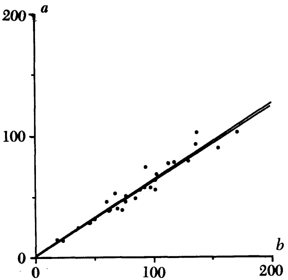


**FIGURE 8**

Scatter plot of $a$ against $b$ with the line $y = \varphi x$ (bottom) and the regression line (top).

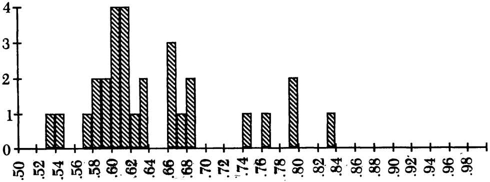


**FIGURE 9**

Frequency distribution of $\frac{a}{b}$.

# Further Analysis

- In this case there is still a reasonably good linear fit ($r^2 = 0.938$, a slight decrease) and the best linear fit is still pretty close to that of the golden ratio.
- But the histogram is nowhere near as impressive. Data does not tend to cluster around a central point given by $1/\phi$.
  How is this possible?
- Nice fact (J. Putz):
  $$\left|\frac{b}{a+b} - \frac{1}{\phi}\right| \leq \left|\frac{a}{b} - \frac{1}{\phi}\right| \quad \text{for any } 0 \leq a \leq b.$$
- This has to do with the fact that $1/\phi$ is an attracting fixed point for the dynamical system determined by $f(x) = 1/(x+1)$.
- Punchline: When checking for the existence of the golden ratio, analyze the ratio of the smallest piece to the largest. The ratio of the largest segment to the whole is always biased toward $1/\phi$.