Math/Music: Aesthetic Links
Group Theory

Definition: The set G is a **group** under the operation $*$ if the following four properties are satisfied:

1. Closure: If $a \in G$ and $b \in G$, then $a * b \in G$. This must be true for all elements a and b in the group G.
2. Associativity: $(a * b) * c = a * (b * c)$
3. Identity: There must exist an element $e \in G$ called the **identity element** such that $a * e = a$ and $e * a = a$. (e preserves the "identity" of the element it is being multiplied by.)
4. Inverse: For every element $a \in G$, there must exist an element $a^{-1} \in G$ called the **inverse of** a, such that $a * a^{-1} = e$ and $a^{-1} * a = e$. Note that the inverse of each element must be in the group G.

Some Examples

- $G = \mathbb{Z}$ (the integers) with $* = +$, the usual addition of two integers. In this case, the identity element is $e = 0$ and $a^{-1} = -a$ since $a * a^{-1} = a + (-a) = 0 = e$.
- $G = \mathbb{R} - \{0\}$ (all real numbers except for 0) with $* = \times$, the usual multiplication of two real numbers. Here, $e = 1$ and $a^{-1} = 1/a$. Why did we have to exclude 0 from G?

  Note: The set $G = \mathbb{Z}$ (the integers) is **not** a group under multiplication. How come?
- $G = \{0, 1, 2, \ldots, 10, 11\}$ with $* = +$ mod 12 (modular arithmetic). Here, $e = 0$ and $a^{-1} = 12 - a$ since 12 is equivalent to 0 in this group ($13 \equiv 1, 14 \equiv 2$, etc.). This group is identical to the group of twelve notes in a chromatic scale. When musicians identify a note with the same note in a different octave, they are doing group theory!
- **Symmetries of the Square: D_4, the Dihedral Group of Degree 4**

  The eight possible symmetries of the square form a group with $*$ = composition.

$*$	e	R_{90}	R_{180}	R_{270}	H	V	D_{13}	D_{24}
e	e	R_{90}	R_{180}	R_{270}	H	V	D_{13}	D_{24}
R_{90}								
R_{180}								
R_{270}								
H	H	D_{13}	V	D_{24}	e	R_{180}	R_{90}	R_{270}
V								
D_{13}								
D_{24}								

Table 1: Multiplication table for the 8 symmetries of the square. Fill this out for HW#1, question 8. Two rows have already been completed for you.