Math/Music: Aesthetic Links

Change Ringing

Rules for ringing an EXTENT on n bells:

1. The first and last changes (rows) are *rounds* ($1\ 2\ 3\ 4 \cdots n$).
2. Other than rounds, all of the other $n!$ changes occur exactly once.
3. Between successive changes, no bell moves more than one position.
4. No bell rests for more than 2 (sometimes relaxed further to 4) changes.
5. Each working bell should do the same amount of "work" (obey the same overall pattern).
6. Horizontal symmetry should be present in the extent to help the ringers learn the path of their respective bell. This is called the *palindrome property*.

Note: Rules 1 - 3 are mandatory for an extent while Rules 4 - 6 are optional though often satisfied.

n	$n!$	Approximate Duration	Name
3	6	15 secs.	*Singles*
4	24	1 mins.	*Minimus*
5	120	5 mins.	*Doubles*
6	720	30 mins.	*Minor*
7	5,040	3 hrs.	*Triples*
8	40,320	24 hrs.	*Major*
9	362,880	9 days	*Caters*
10	3,628,800	3 months	*Royal*
11	39,916,800	3 years	*Cinques*
12	479,001,600	36 years	*Maximus*

Table 1: Approximate duration to ring an extent on n bells and the names given to such an extent. For example, *Plain Bob Minimus* is a composition written for 4 bells while *Grandshire Triples* is an extent on 7 bells.

The two extents on 3 bells:

1 2 3	**1** 2 3
2 **1** 3	**1** 3 2
2 3 **1**	3 **1** 2
3 2 **1**	3 2 **1**
3 **1** 2	2 3 **1**
1 3 2	2 **1** 3
1 2 3	**1** 2 3

Note the simple zig-zag pattern of Bell **1** in the first extent, sweeping easily from position 1 to position 3 and back again. We say that Bell **1** is *plain hunting.* It only needs to do this once to complete the extent. In this case, we say that the bell is “not working.” Notice that in the second extent, Bell **1** follows a similar zig-zag path except that this begins on the second change.

Plain Bob Minimus (read down first, then hop to next column)

1 2 3 4	**1** 3 4 2	**1** 4 2 3
2 **1** 4 3	3 **1** 2 4	4 **1** 3 2
2 4 **1** 3	3 2 **1** 4	4 3 **1** 2
4 2 3 **1**	2 3 4 **1**	3 4 2 **1**
4 3 2 **1**	2 4 3 **1**	3 2 4 **1**
3 4 **1** 2	4 2 **1** 3	2 3 **1** 4
3 **1** 4 2	4 **1** 2 3	2 **1** 3 4
1 3 2 4	**1** 4 3 2	**1** 2 4 3
		1 2 3 4

Note the similarities with the first extent on three bells above. Bell **1** goes plain hunting again, this time moving from position 1 to position 4 and back again, needing 3 cycles to complete the extent. The other three bells have the same paths, just starting at different places so that Rule 5 is satisfied. This is a bit like a round. Letting $a = (1\,2)(3\,4), b = (2\,3)$ and $c = (3\,4)$ be the three permutations used in the extent, Plain Bob Minimus can be written (factored) as $[(ab)^3ac]^3$. Since *abababa* is a palindrome, Rule 6 is satisfied for this extent. There are lots of interesting patterns and mathematics lurking in this extent!

Canterbury Minimus (read down first, then hop to next column)

1 2 3 4	**1** 3 4 2	**1** 4 2 3
2 **1** 4 3	3 **1** 2 4	4 **1** 3 2
2 4 **1** 3	3 2 **1** 4	4 3 **1** 2
2 4 3 **1**	3 2 4 **1**	4 3 2 **1**
4 2 3 **1**	2 3 4 **1**	3 4 2 **1**
4 2 **1** 3	2 3 **1** 4	3 4 **1** 2
4 **1** 2 3	2 **1** 3 4	3 **1** 4 2
1 4 3 2	**1** 2 4 3	**1** 3 2 4
		1 2 3 4

What are the similarities and differences with Plain Bob Minimus? Is this a legitimate extent? Which of the six rules does it satisfy?

Some More Group Theory

Consider the extent Plain Bob Minimus as a list of all permutations in the group S_4.

$e =$	**1** 2 3 4	**1** 3 4 2	**1** 4 2 3
$\alpha =$	2 **1** 4 3	3 **1** 2 4	4 **1** 3 2
$\beta =$	2 4 **1** 3	3 2 **1** 4	4 3 **1** 2
$\alpha\beta =$	4 2 3 **1**	2 3 4 **1**	3 4 2 **1**
$\beta^2 =$	4 3 2 **1**	2 4 3 **1**	3 2 4 **1**
$\alpha\beta^2 =$	3 4 **1** 2	4 2 **1** 3	2 3 **1** 4
$\beta^3 =$	3 **1** 4 2	4 **1** 2 3	2 **1** 3 4
$\beta\alpha =$	**1** 3 2 4	**1** 4 3 2	**1** 2 4 3
			1 2 3 4

We label the first two changes after rounds as α and β (α and β are pronounced "alpha" and "beta" respectively. These are the first two letters of the Greek alphabet, commonly used in mathematics.) One can check that the remaining 5 changes of the first lead are all expressible in terms of α and β, given by the formulas shown above. Note that we are dropping the $*$ here for ease of notation, so for example, $\alpha\beta = \alpha * \beta$. Remember that $\alpha\beta$ means we apply the permutation α first, then take the result and apply β. The order often matters!

On HW#2, one of the goals is to prove that the first column (lead) of Plain Bob Minimus forms a group under multiplication of permutations (call it Set A). This group contains 8 elements and is a **subgroup** of the larger group S_4. The hardest thing to show is that the elements in Set A are closed under multiplication of permutations. In other words, the product of any two permutations in the first column gives you a permutation that is still in the first column. This can be accomplished by making a multiplication table and checking that all 64 products are indeed elements of the first lead.

Instead of doing every multiplication out by hand, there are some identities involving α and β which are particularly useful. In turn, these identities can be used to derive other useful relations. You might keep a list of them as you fill out your table. Here are some key ones:

$$\alpha^2 = e$$
$$\beta^4 = e$$
$$\beta\alpha\beta = \alpha$$

Using the Identities

Suppose we wanted to find the product $\alpha\beta^3$ using the identities above. Multiplying **on the right** of each side of the last identity gives us

$$\beta\alpha\beta * \beta^3 = \alpha * \beta^3$$

The left-hand side of this equation simplifies to $\beta\alpha\beta^4 = \beta\alpha e = \beta\alpha$ which is listed in the fifth row of Table 2. It is very important that you multiply the same way on each side of the equation. Since $*$ is not usually commutative, the order matters! The goal is for you to fill out the multiplication table on the next page, **obtaining only elements from our supposed subgroup Set A.**

*	e	β	β^2	β^3	α	$\alpha\beta^2$	$\alpha\beta$	$\beta\alpha$
e	e	β	β^2	β^3	α	$\alpha\beta^2$	$\alpha\beta$	$\beta\alpha$
β								
β^2								
β^3								
α	α	$\alpha\beta$	$\alpha\beta^2$	$\beta\alpha$	e	β^2	β	β^3
$\alpha\beta^2$								
$\alpha\beta$								
$\beta\alpha$								

Table 2: Multiplication table for the 8 permutations in the first lead of Plain Bob Minimus. Fill this out for HW#2. Two rows have already been completed to help you get started. How does this table compare with the table you constructed for the dihedral group D_4, the symmetries of the square?

Mathematical Side Note

It is tempting to ask whether the other leads in Plain Bob Minimus are also subgroups. The answer is no since neither contains rounds, which is the identity element, so property 3 for groups does NOT hold. However, it is easy to generate these leads from the subgroup formed by the first lead. Multiplication by the permutation (1 3 4 2) on the right takes the entire first column to the second column and multiplication again by this same permutation takes the second column to the third. In group theory, the second and third leads are called **cosets**. A coset is obtained from a subgroup by multiplying on the left or the right every element in the subgroup. Thus we can speak of left or right cosets. Each coset has the same number of elements as the subgroup just as our three leads each have the same number of changes (eight). It turns out that many extents have this decomposition where the first lead is a subgroup and the remaining leads are just cosets generated by this group. Mathematician, composer and change ringer Arthur White acknowledges this fact in the title of his wonderful paper on change ringing, *Ringing the Cosets*, American Mathematical Monthly, Oct. 1987. Moreover, the elements used in generating the other leads form a different subgroup together, called a **cyclic subgroup of order** n. This is a group of the form $\{\omega, \omega^2, \omega^3, \ldots, \omega^n = e\}$. It is clear that there is a great deal of group theory involved in doing change ringing!