

# Math/Music: Aesthetic Links

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Musical Group Theory  
Symmetry in Music  
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## Symmetry in Music: Group Theory

How to get more music out of a little motif:

Translations (shifting graph vertically)  $\iff$  Transpositions (shifting notes up or down)

Ex: Ballpark Music

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Retrograde (music same forward and backward)

Ex: Lean on Me

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How to get more music out of a little motif:

Translations (shifting graph vertically)  $\iff$  Transpositions (shifting notes up or down)

Ex: **Ballpark Music**

Vertical Reflection (symmetry between right and left)  $\iff$   
Retrograde (music same forward and backward)

Ex: **Lean on Me**

Horizontal Reflection (symmetry between top and bottom)  $\iff$   
Inversion (what goes up, must come down)

Ex: **Bach, Bach and more Bach**

# Symmetry in Music: Retrograde

MENUET AL ROVESCIO

The image displays a musical score for a minuet in G major, titled 'Menuet al Rovescio'. The score is presented in four systems, each with a grand staff (treble and bass clefs). The first system shows the original melody in the treble clef and its accompaniment in the bass clef. The second system continues the original piece. The third system begins with a double bar line and repeat signs, indicating the start of the retrograde section. The fourth system shows the original melody in the bass clef and its accompaniment in the treble clef, demonstrating the piece played backwards. The key signature is one sharp (F#) and the time signature is 3/4.

Figure: Joseph Haydn, *Piano Sonata No. 41, Hob. XVI/26*, Minuet



## Symmetry in Music: Retrograde

The image displays five systems of musical notation for Johann Sebastian Bach's *A Musical Offering*. Each system consists of two staves: a treble clef staff on top and a bass clef staff on the bottom. The music is written in G major (one sharp) and 3/4 time. The first system shows the beginning of the piece. The second system continues the melody. The third system features a more complex rhythmic pattern. The fourth system shows a continuation of the melody. The fifth system concludes the piece with a final cadence. The notation includes various note values, rests, and bar lines, illustrating the intricate structure of the piece.

Figure: Johann Sebastian Bach, *A Musical Offering*

## Symmetry in Music: Inversion



for the Lord God Om - ni - po - tent reign - eth



George F. Handel, *Messiah*, *Hallelujah chorus* (lose retrograde, form of tone painting)

**Allegro**

Musical notation for the beginning of Mikrokosmos No. 141. It is a piano piece in 3/4 time, marked Allegro. The notation is for the first system, showing the right and left hands. The right hand starts with a treble clef and the left hand with a bass clef. The key signature has two flats (B-flat and E-flat). The melody in the right hand is: G4 (quarter), A4 (quarter), B4 (quarter), C5 (quarter), B4-A4 (beamed eighth notes), G4 (quarter), F#4 (quarter), E4 (quarter), D4 (half). The left hand provides a harmonic accompaniment with chords and moving lines.

Béla Bartók, *Mikrokosmos*, No. 141, *Subject and reflection* (inversion)

## Symmetry in Music: Rotation



—and here follows an hour of music—



Figure: Paul Hindemith, *Ludus Tonalis* (“Tonal Game”), beginning and end. The ending Postludium is an **exact** retrograde-inversion of the opening Praeludium.

# Bach: The Well-Tempered Clavier, Fugue No. 8 in D $\sharp$ minor

Handbook for Keyboard Teacher + Performer Chapter Five A Proposed Course of Study and Analysis  
by Laurette Adberg Example 5.2 Fugue No. 8 in d $\sharp$  minor from WTC I by J.S. Bach ABA' form (Sonata-form)

Exposition 3 Tonal answer

6 8

11 1a (redundant statement)

15 Development

20 (rhythmic sync.)

24 (rhythmic sync.)

25 S S S



60 <sup>6a</sup> **Recapitulation**

chrom. variation

S-augmented

64 S-inverted F# Major S-aug.

69 S

73 Coda (stretto) S-aug. S

79 rhythmic orn. (orn.) S (orn.)

83

John Philip Sousa "The Thunderer"



Franz Liszt "Hungarian Rhapsody # 2"

THE MATH BEHIND THE MUSIC



Fig. 11

# Combining Symmetries

9a.

1 2 3 4    2 2 3 4    3 2 3 4    4 2 3 4

B $\flat$     B $\flat$ 6    C $m$ 7    F7    B $\flat$     E $^{\circ}$ 7    C $m$ 7

**"I Got Rhythm" has an AABA structure, and a two-bar tag at the end.** We call these four equal sections A, A, B, and A. Three of the sections are the same, and one is different. Listen to the first section, and you'll be able

9b.

[A] B $\flat$  B $\flat$ 6 C $m$ 7 F7 B $\flat$ 6 E $^{\circ}$ 7 C $m$ 7 F7 B $\flat$  B $\flat$ 6 C $m$ 7 F7 E $b$ m6

[A] B $\flat$  F7 B $\flat$  B $\flat$  B $\flat$ 6 C $m$ 7 F7 B $\flat$ 6 E $^{\circ}$ 7 C $m$ 7 F7 B $\flat$  B $\flat$ 6

[B] Bridge C $m$ 7 F7 E $b$ m6 B $\flat$  F7 B $\flat$  D7 A $m$ 7 F $m$ 6 D7 G D+ G9 G7

[A] C7 G $m$ 7 E $b$ m6 C9 C7-5 F7 B $\flat$  B $\flat$ 6 C $m$ 7 F7 B $\flat$ 6 E $^{\circ}$ 7

Tag C $m$ 7 F7 B $\flat$  B $\flat$ 6 C $m$ 7 F7 E $b$ m B $\flat$  F $m$  G7 C7 F7 B $\flat$

**Figure:** George Gershwin, *I Got Rhythm*, (transposition, retrograde and inversion, all in one song!)

# Der Spiegel (The Mirror) Duet

VOLTA 1 *Allegro*  $J=120$

W.A. Mozart

*mf*

*Allegro*

Public Domain. Synchronized by Fred Nachbauer using NoteWorthy  
Codemore? Try playing this from opposite sides of a table.

(Note: the attribution to Mozart is dubious)

Figure 9.6.