Math/Music: Aesthetic Links

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Composing with Numbers:
Arnold Schoenberg and His Twelve-Tone Method
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Chapter 8: Composing with numbers: Sets, rows and magic squares

by Jonathan Cross

- Arnold Schoenberg (1874-1951, Austro-Hungarian), one of the most influential 20th century composers
- Pioneer in atonal music (no key, no tone, no "tonality")
  "Tonality is not an eternal law of music, but simply a means toward the achievement of musical form."
- Invented the twelve-tone method of composing. Start with a particular tone row — the twelve notes of the chromatic scale arranged in some particular order, with each note occurring exactly once — and build all of the music by applying different musical symmetries to the tone row (transposition, inversion, retrograde and retrograde-inversion).

The Twelve-Tone Method of Composition

Notation:

1. P-n: transpose the original tone row up n half-steps (translation)
2. R-n: retrograde of P-n (vertical reflection)
3. I-n: inversion (exact) of P-n about its starting note (horizontal reflection). This is equivalent to transposing I-0 up n half-steps.
4. RI-n: retrograde-inversion of P-n (180° rotation). This is equivalent to taking the retrograde of I-n.

In general, this gives 48 possible variants of a tone row, 12 of each type. P-12 = P-0, P-13 = P-1, P-14 = P-2, etc. (arithmetic mod 12)

The octave a pitch appears in is irrelevant. In other words, all C's on the piano are considered the same, all C♯'s are the same, etc. We are really manipulating pitch classes.

Initially, music critics thought this was the compositional equivalent of "painting by numbers."

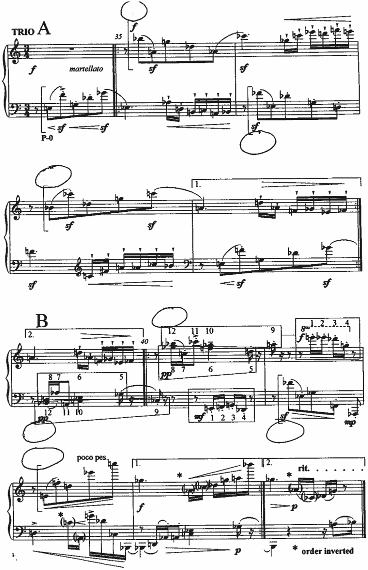

Figure: Arnold Schoenberg, *Piano Suite, Op. 25* (1921-23). Trio from the Minuet and Trio (without tone-row labeling).

Schoenberg's 6 Tone Rows

Figure: Tone rows and their symmetries.

Analysis of Schoenberg's Piano Suite, Op. 25

- Same original or prime tone row used for basis of entire piece.
- In the Trio, only the six rows P-0, P-6, I-0, I-6, R-6 and RI-6 are used.
- The tritone interval (six half-steps) is emphasized since only $n = 0$ or $n = 6$ is utilized. The primary row begins on an E and ends on a B$\flat$, which is also a tritone. This means that all six rows begin and end on either an E or a B$\flat$, providing some continuity to the music.
- Trio is structurally a canon, with the right hand imitating the left rhythmically (exact) after one full measure (three beats).

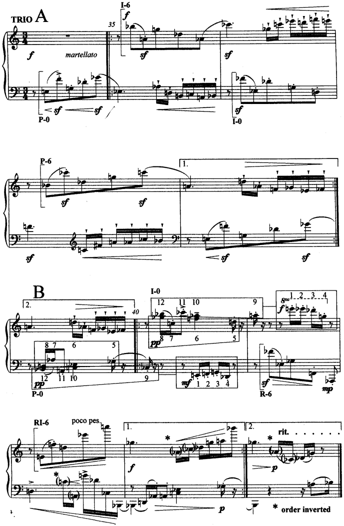

Figure: Arnold Schoenberg, *Piano Suite, Op. 25*. Trio from the Minuet and Trio (tone row labels included).

Key Observation:

Not all primary tone rows generate 48 different rows. It is possible that different operations may generate the same row, particularly when the prime row has some type of symmetry.

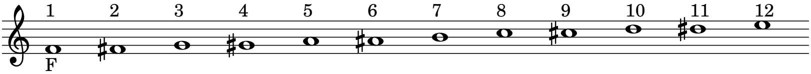

Figure: P-0 is a chromatic scale beginning on F.

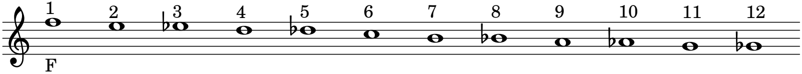

Figure: The tone row I-0 is a descending chromatic scale beginning on F.

But R-0 is a descending chromatic scale starting on E, R-1 is a descending chromatic scale starting on F, etc.

Thus: I-0 = R-1, I-1 = R-2, I-2 = R-3, ... , I-10 = R-11, I-11 = R-0.

Tone Rows and Symmetry (cont.)

Moreover, applying the retrograde operation to both sides of each equation in the list

I-0 = R-1, I-1 = R-2, I-2 = R-3, ... , I-10 = R-11, I-11 = R-0

yields

RI-0 = P-1, RI-1 = P-2, RI-2 = P-3, ... , RI-10 = P-11, RI-11 = P-0

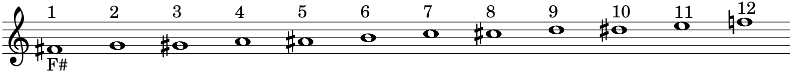

Figure: The tone row RI-0 is an ascending chromatic scale beginning on F♯.

Conclusion: For this particular prime tone row, there are only 24 unique rows generated by the symmetry operations, not 48. In other words, there are only 24 different chromatic scales, 12 ascending and 12 descending.

More Group Theory

Question: How likely is it that a tone row will only generate 24 new rows, rather than 48?

Total number of possible tone rows: $12! = 479,001,600$

Total number of prime rows that are fixed under some symmetry operation: $7 \cdot (12 \cdot 10 \cdot 8 \cdot 6 \cdot 4 \cdot 2) = 322,560$.

Answer: Only a 0.06734% chance of this happening.

If you were wondering ...

Group Structure: The 48 symmetry operations P-n, R-n, I-n, RI-n form a group G that is isomorphic to $D_{12} \times \mathbb{Z}_2$. If you call two tone rows the "same" if they are equivalent under some operation of G, then there are precisely 9,985,920 "different" tone rows (David Reiner). Proof uses group theory and Burnside's counting lemma.