Math/Music: Aesthetic Links

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The Fibonacci Numbers and the Golden Ratio
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The Fibonacci Numbers

Definition

The Fibonacci Numbers are the numbers in the sequence

$$1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, \ldots \; .$$

This is a recursive sequence defined by the equations

$$F_1 = 1, \; F_2 = 1, \quad \text{and} \quad F_n = F_{n-1} + F_{n-2} \quad \text{for all } n \geq 3.$$

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Often called the "Fibonacci Series" or "Fibonacci Sequence".

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- Imported the Hindu-arabic decimal system to Europe in his book *Liber Abbaci* (1202). Latin translation: “book on computation.”

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- Recursive pattern is clear: To find the number of ways to subdivide n beats, take all the possibilities for $n - 2$ beats and append an L, and take those for $n - 1$ and append an S.

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- The rap group *Black Star* uses the following lyrics in the song "Astronomy (8th Light)"

  Now everybody hop on the one, the sounds of the two
  It's the third eye vision, five side dimension
  The 8th Light, is gonna shine bright tonight

Fibonacci Numbers in the Comics

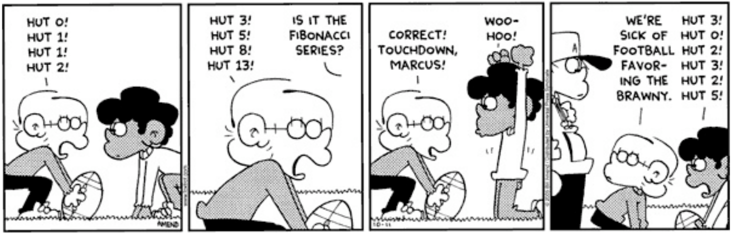

Figure: *FoxTrot* by Bill Amend (2005)

The Rabbit Problem

Key Passage from the 3rd section of Fibonacci's *Liber Abbaci*:

> *"A certain man put a pair of rabbits in a place surrounded on all sides by a wall. How many pairs of rabbits can be produced from that pair in a year if it is supposed that every month each pair begets a new pair which from the second month on becomes productive?"*

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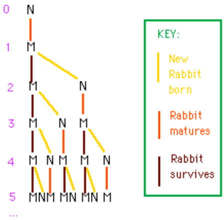

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	Parents	Gr-parents	Gt-Gr-parents	Gt-Gt-Gr-p's	G-G-G-G-p's
D	1	2	3	5	8
Q	2	3	5	8	13

Table: Number of parents, grand-parents, great-grand parents, etc. for a drone and queen bee.

Fibonacci Numbers in Nature

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- Number of leaves in one full turn around the stem of some plants.
- This is not a coincidence! Some of the facts about spirals can be explained using continued fractions and the golden mean.

Figure: Columbine (left, 5 petals); Black-eyed Susan (right, 13 petals)

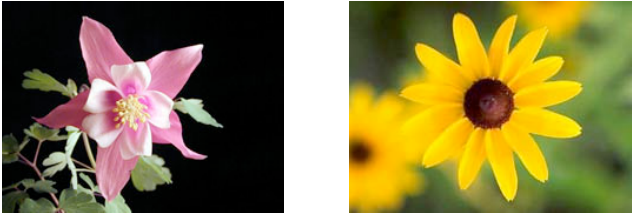

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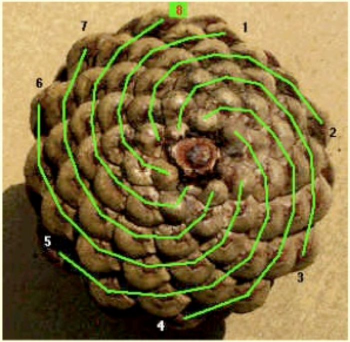
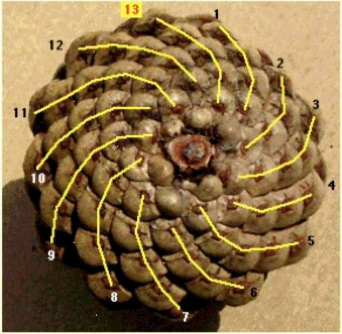
Figure: Shasta Daisy (left, 21 petals); Field Daisies (right, 34 petals)

Bracts arranged in Fibonacci numbers of spirals



Adjacent Fibonacci numbers, 8, 13



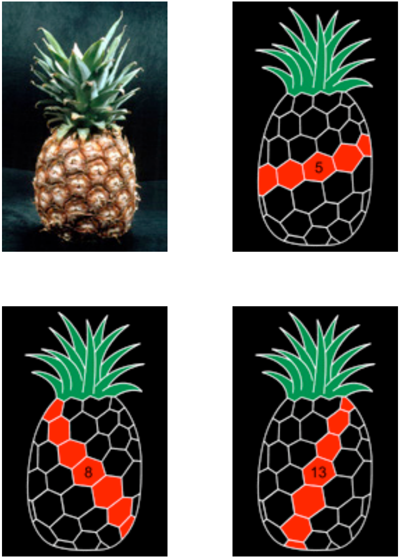
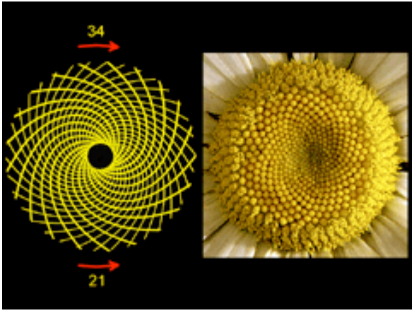


Figure: Pineapple scales often have three sets of spirals with 5, 8 and 13.



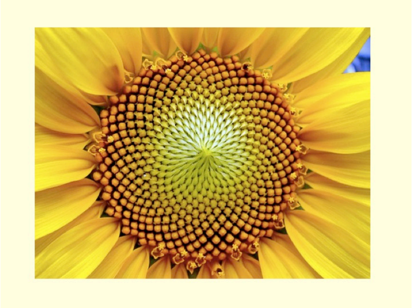
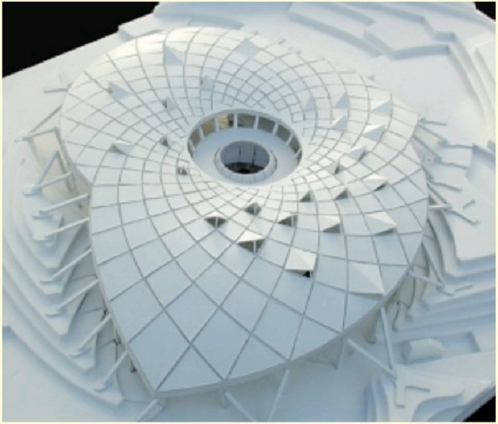


Figure: In most daisy or sunflower blossoms, the number of seeds in spirals of opposite direction are consecutive Fibonacci numbers.



Model of the Education Building, The Eden Project, Cornwall
by Joylon Brewis and Peter Randall-Page

Figure: The chimney of Turku Energia in Turku, Finland, featuring the Fibonacci sequence in 2m high neon lights (Mario Merz, 1994).

Figure: Structure based on a formula connecting the Fibonacci numbers and the golden mean. The fountain consists of 14 (?) water cannons located along the length of the fountain at intervals proportional to the Fibonacci numbers. It rests in Lake Fibonacci (reservoir).

A Nice Geometric Proof

Fibonacci Identity 5(d):

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$$1^2 + 1^2 + 2^2 + 3^2 + 5^2 + 8^2 \;=\; 104 \;=\; 8 \cdot 13 \;=\; F_6 \cdot F_7$$

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Geometric Proof: Start with a 1×1 square. Place another 1×1 square above it, and then place a 2×2 square to its right. Place a 3×3 square below the preceding blocks, and a 5×5 square to the left. Place an 8×8 square above and continue the process of placing squares in a clockwise fashion.

A Nice Geometric Proof (cont.)

At the *n*th stage in the process you will have constructed a rectangle whose area is the sum of all the squares:

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On the other hand, the area of a rectangle is just length times width, which in this case is

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This proves the identity

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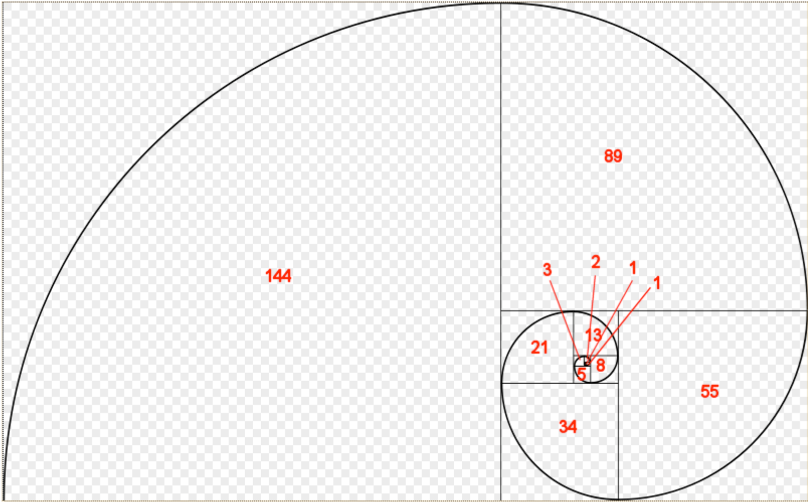


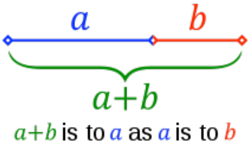
Figure: The Fibonacci Spiral, which approximates the Golden Spiral, created in a similar fashion but with squares whose side lengths vary by ϕ. Each are examples of Logarithmic Spirals, very common in nature.

Chambered nautilus shell

Figure: The Pinwheel Galaxy (also known as Messier 101 or NGC 5457).

Fibonacci wall art, by Dan Freund

Connections with the Golden Ratio



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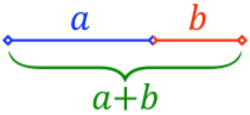

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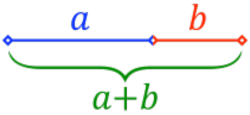

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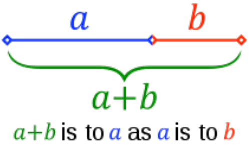

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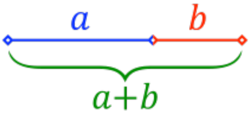

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Note: This is true for any recursive sequence with $F_n = F_{n-1} + F_{n-2}$, not just the Fibonacci sequence.

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- Named ϕ in the mid-20th century in honor of the ancient Greek architect Phidias.

The Pentagram

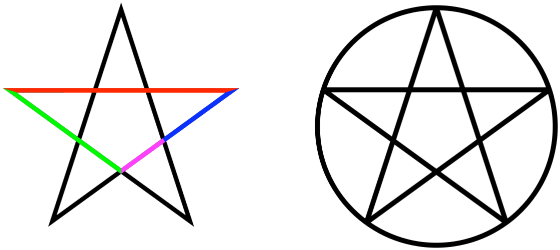

Figure: Left: the Pentagram — each colored line segment is in golden ratio to the next smaller colored line segment. Right: the Pentacle (a pentagram inscribed inside a circle.)

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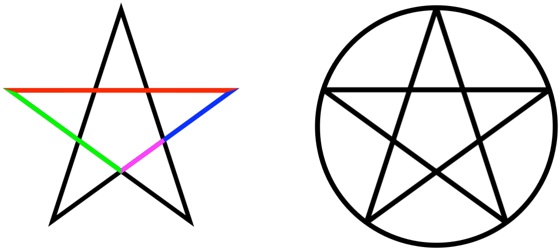

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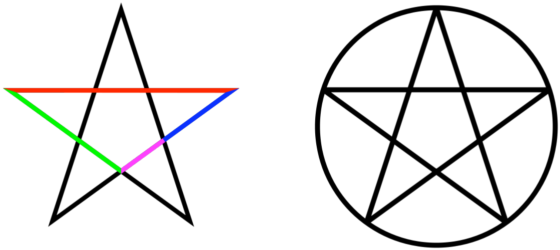

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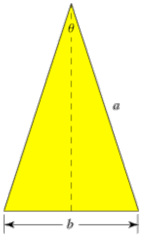
The Pythagoreans used the Pentagram (called it *Hugieia*, “health”) as their symbol in part due to the prevalence of the golden ratio in the line segments.
The pentagram is a five-pointed star that can be inscribed in a circle with equally spaced vertices (regular pentagon).

The Golden Triangle



Note: The isosceles triangle formed at each vertex of a pentagram is a golden triangle. This is an isosceles triangle where the ratio of the hypotenuse a to the base b is equal to the golden ratio $\phi = a/b$.

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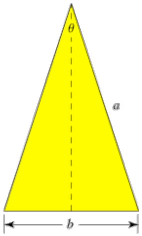

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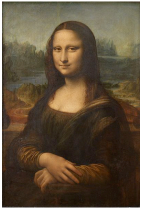
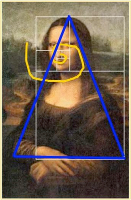
Using some standard trig. identities, one can show that

$$\theta \;=\; 2\sin^{-1}\left(\frac{b}{2a}\right) \;=\; 2\sin^{-1}\left(\frac{1}{1+\sqrt{5}}\right) \;=\; \frac{\pi}{5} \;=\; 36^\circ.$$

The two base angles are then each $2\pi/5 = 72^\circ$.

The Divine Proportion





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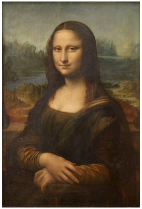

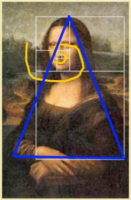

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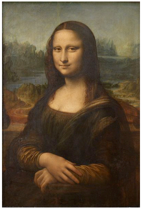

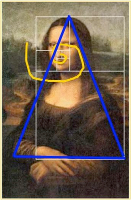

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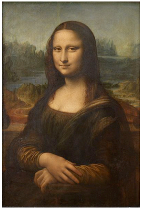 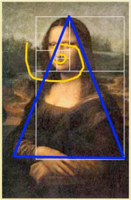

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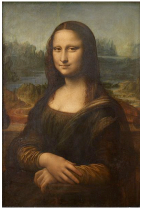

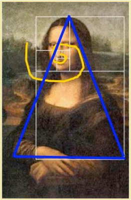

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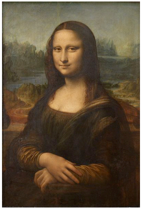

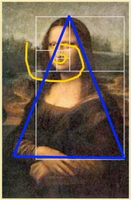

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Luca Pacioli's *De Divina Proportione* (1509) was illustrated by Leonardo da Vinci (Pacioli was his math teacher), demonstrating ϕ in various manners (e.g., architecture, perspective, skeletonic solids).

Fibonacci Phyllotaxis

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Question: How come so many plants and flowers have Fibonacci numbers?

Succint Answer: Nature tries to optimize the number of seeds in the head of a flower. Starting at the center, each successive seed occurs at a particular angle to the previous, on a circle slightly larger in radius than the previous one. This angle needs to be an irrational multiple of 2π, otherwise there is wasted space. But it also needs to be poorly approximated by rationals, otherwise there is still wasted space.

Fibonacci Phyllotaxis (cont.)

Figure: Seed growth based on different angles α of dispersion. Left: $\alpha = 90°$. Center $\alpha = 137.6°$. Right: $\alpha = 137.5°$.

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What is so special about 137.5°? It's the golden angle!

Dividing the circumference of a circle using the golden ratio gives an angle of

$$\alpha \;=\; \pi(3 - \sqrt{5}) \;\approx\; 137.5077641°.$$

This seems to be the best angle available.

Example: The Golden Angle

Figure: The Aonium with 3 CW spirals and 2 CCW spirals. Below: The angle between leaves 2 and 3 and between leaves 5 and 6 is very close to 137.5°.

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Wow! Mother Nature Knows Math.