

Math/Music: Aesthetic Links

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Composing with Numbers:
Sir Peter Maxwell Davies and Magic Squares
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Sir Peter Maxwell Davies

- Born in Salford, Lancashire England, 1934.
- Studied at University of Manchester and Royal Manchester College of Music. Helped form the group *New Music Manchester*, committed to contemporary music.
- Studied in Rome, Princeton and Australia before settling on the Orkney Islands (recall *Collapse*).
- Served as the associate conductor/composer with the BBC Philharmonic Orchestra from 1992-2002.
- Has conducted major orchestras such as the Cleveland and Boston Symphony Orchestras.

Sir Peter Maxwell Davies (cont.)

- Knighted in 1987 (thus the Sir honorific).
- In 2004 appointed Master of the Queen's Music.
- Awarded Honorary Doctorate of Music by Oxford University in 2005.
- One of the first classical composers to open a music download website `maxopus.com` (1996) promoting his works.
- Known for using **magic squares** as an architectural blueprint for creating structures in his music, even generating the music itself.
- Example: *Ave Maris Stella* (1975): 9×9 magic square associated with the moon to permute notes of a plainchant melody and to determine durations of the notes.

Magic Squares

Definition

A square array of numbers where each row, column and main diagonal sums up to the same amount is called a **magic square**. An $n \times n$ magic square uses only the numbers $1, 2, 3, 4, \dots, n^2 - 2, n^2 - 1, n^2$.

An $n \times n$ **matrix** is a square array of numbers with n rows and n columns.

Example: ($n = 3$)

$$\begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

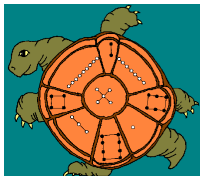
This is not a magic square.

The case 3×3

8	1	6
3	5	7
4	9	2

- Each row sums to 15.
- Each column sums to 15.
- Each of the two main diagonals sums to 15.
- MAGIC!!!
- The number 15 is called the **magic constant** (particular to $n = 3$.)
- Up to rotation and reflection, (think D_4), this is the **only** possible 3×3 magic square.
- The square was known to the ancient Chinese (over 3,000 years ago), who called it **Lo Shu**.

History of the Lo Shu Magic Square



- Huge flood – one of the rivers flooding was the **Lo** river.
- People tried to make some sacrifice to the river god, but didn't know how much to sacrifice.
- Turtles emerged from the river but always circled around the sacrifice.
- Eventually, a special turtle emerged with the special markings above. A child noticed the pattern of 15 and then the people knew the correct sacrifice to make.
- Feng Shui formulas of astrology, flying star and I-Ching are all based on this Lo Shu magic square.
- **Shu** means book or scroll.

Dürer's 4×4 Magic Square

16	3	2	13
5	10	11	8
9	6	7	12
4	15	14	1

- The sum of any row, column or main diagonal is 34. Magic!!
- 1514 engraving [Melancholia I](#) by German artist (painter, printmaker, engraver) and mathematician Albrecht Dürer featured the above magic square.
- The date of the engraving appears in the middle of the bottom row of the magic square.
- Up to symmetry, there are a total of 880 "different" 4×4 magic squares.



Figure: Dürer's Melancholia I, 1514

What about a 2×2 magic square?

Try $n = 2$.

1	2
3	4

1	2
4	3

1	3
2	4

1	3
4	2

1	4
2	3

1	4
3	2

Conclusion: There is no 2×2 magic square.

The Magic Constant

Nice Fact: (HW) The rows, columns and main diagonal of **any** $n \times n$ magic square sum up to the same number M_n , called the **magic constant**. This number changes (increases) with n and is given by

$$M_n = \frac{n(n^2 + 1)}{2}.$$

n	M_n
2	5
3	15
4	34
5	65
6	111
7	175
8	260
9	369
10	505

A Mirror of Whitening Light

- Composed by Davies in 1977.
- Latin title: “Speculum Luminis de Albensis”.
- Commissioned by the [London Sinfonietta](#) (instrumental ensemble).
- Dedicated to the American composer [Roger Sessions](#) (one of his teachers).
- Title refers to alchemy, in particular the process of “whitening” a base metal into gold.
- Inspired by the remarkable properties of light, as noticed while working in Orkney.
- Uses the 8×8 magic square of Mercury to create entire set of notes and durations for the piece.

The Architectural Blueprint

8	58	59	5	4	62	63	1
49	15	14	52	53	11	10	56
41	23	22	44	45	19	18	48
32	34	35	29	28	38	39	25
40	26	27	37	36	30	31	33
17	47	46	20	21	43	42	24
9	55	54	12	13	51	50	16
64	2	3	61	60	6	7	57

Table: The 8×8 magic square of Mercury.

Davies takes 8 notes from the plainchant **Veni Sancte Spiritus** (Come Holy Spirit), often called the “Golden Sequence,” to create a melodic phrase with 8 distinct notes. He then transposes the phrase to start on each of the 8 notes, yielding an 8×8 matrix of notes. This matrix is then mapped onto the magic square of Mercury. Different paths through the resulting magical music square generate the pitches and rhythms for the piece.

(b)

ve - ni Sanc - te Spi - ri - tus re - ple tu - o - rum cor - de fi - de - li - um etc.

Summary (derived from the plainchant and retaining its profile)

G ₁	E ₂	F ₃	D ₄	F [#] ₅	A ₆	G [#] ₇	C ₈
E ₉	C [#] ₁₀	D ₁₁	B ₁₂	D [#] ₁₃	F [#] ₁₄	F ₁₅	A ₁₆
F ₁₇	D ₁₈	E [#] ₁₉	C ₂₀	E ₂₁	G ₂₂	F [#] ₂₃	A [#] ₂₄
D ₂₅	B ₂₆	C ₂₇	A ₂₈	C [#] ₂₉	E ₃₀	D [#] ₃₁	G ₃₂
F [#] ₃₃	D [#] ₃₄	E ₃₅	D [#] ₃₆	F ₃₇	A [#] ₃₈	G ₃₉	B ₄₀
A ₄₁	F [#] ₄₂	G ₄₃	E ₄₄	G [#] ₄₅	B ₄₆	B [#] ₄₇	D ₄₈
G [#] ₄₉	F ₅₀	F [#] ₅₁	E [#] ₅₂	G ₅₃	B [#] ₅₄	A ₅₅	C [#] ₅₆
C ₅₇	A ₅₈	B [#] ₅₉	G ₆₀	B ₆₁	D ₆₂	C [#] ₆₃	F ₆₄

8 × 8 matrix

C ₈	A ₅₈	B [#] ₅₉	F [#] ₅	D ₄	D ₆₂	C [#] ₆₃	G ₁
G [#] ₄₉	F ₁₅	F [#] ₁₄	E [#] ₅₂	G ₅₃	D ₁₇	C [#] ₁₀	C [#] ₅₆
A ₄₇	F [#] ₂₃	G ₂₂	E ₄₄	G ^(D) ₄₅	E [#] ₁₉	D ₁₈	(G [#]) _D ₄₈
G ₃₂	D [#] ₃₄	E ₃₅	C [#] ₂₉	(D) _A ₂₈	A [#] ₃₈	G ₃₉	(A) _D ₂₅
B ₄₀	B ₂₆	C ₂₇	F ₃₇	D [#] ₃₆	E ₃₀	D [#] ₃₁	F [#] ₃₃
F ₁₇	B [#] ₄₇	B ₄₆	C ₂₀	E ₂₁	G ₄₃	F [#] ₄₂	A [#] ₂₄
E ₉	A ₅₅	B [#] ₅₄	B ₁₂	D [#] ₁₃	F [#] ₅₁	F ₅₀	A ₁₆
F ₆₄	E ₂	F ₃	B ₆₁	G ₆₀	A ₆	G [#] ₇	C ₅₇

Magic square of Mercury

Figure 5. Peter Maxwell Davies, *A mirror of whitening light*.

Figure: How Davies uses the Magic Square of Mercury

(c)

The image displays a page of a musical score for Peter Maxwell Davies' 'A mirror of whitening light'. It features 15 staves for various instruments: Flute, Oboe, Clarinet, Bassoon, Horn, Trumpet, Trombone, Cymbals, Celesta, Violin I, Violin II, Viola, Violoncello, and Double bass. The score is marked with 'Presto J = 180' and includes dynamic markings such as *pp* and *ff*. Performance instructions include 'fermata lunga' and 'lasciare vibrare'. Several sections of the score are enclosed in boxes, with lines indicating connections or relationships between these sections across different instruments. The notation includes notes, rests, and articulation marks.

Figure 5. Peter Maxwell Davies, *A mirror of whitening light*.

Figure: Opening to *A Mirror of Whitening Light*

Opening Notes

Trumpet: C A B \flat G \flat D

Flute: D C \sharp G A \flat F G \flat E \flat G

These are, in order, the first 13 pitches of the top two rows.

Clarinet: A F \sharp G E (D) E \flat D (A \flat) G E \flat E C \sharp

Crotales: (antique cymbals) (D) A \flat

Celesta: G (A) B B C F C \sharp E D \sharp F \sharp F B \flat B C

These are (with a few exceptions), in order, the pitches for the next three and a half rows of the magic square.

(d)

Andante $\dot{j} = c.60$ Clarinet Pitches 47 17 33 31 30

Cl.

Bsn

Bassoon Durations

Mar.

Vln I

Vln II

Vcl.

mp

Cor Anglais Durations 1 1 7 ...

Cor Ang.

Cl.

Bsn

Mar.

Vln I

Vln II

Vla

Vcl.

mp

Figure 5. Peter Maxwell Davies, *A mirror of whitening light*.

Figure: An example of durations based on the magic square.

The Magic Square and Arithmetic Mod 8

In this excerpt, the clarinet pitches are obtained by beginning at square 47 (6th row, 2nd column) and tracing through the magic square backwards.

47 17 33 31 30 36 37 27 26 40 ...

Meanwhile, the lengths of the notes in the bassoon part are obtained by taking the same numbers from the clarinet part, computing their remainder upon division by 8 (mod 8), and then assigning 1 = an eighth note, 2 = two eighths (one quarter), 3 = three eighths (dotted quarter), etc.

7 1 1 7 6 4 5 3 2 0(=8) ...

However, the bassoon rhythms are written in reverse order from the clarinet pitches, or forwards through the magic square mod 8.

Davies Describing his Piece

“And if you go across that square of the numbers arranged in a particular way, they make very interesting patterns. And I see these patterns, in the first place, possibly as dance patterns; and one gets to know them by heart. One doesn’t in fact deal with numbers at all. One deals rather as somebody who is dealing with *bell-changes*, with actual patterns with changes” (emphasis added).

Rigid structure imposed by the magic square?

“I firmly believe that the more one controls the flow of one’s wildest inspiration, the wilder it sounds. And so when I really wanted to be wild towards the climax of this work, I imposed very rigid rhythmic and tonal controls derived from the plainsong, and from that magic square; and the result is really quite extraordinary I find, even now.”