Math/Music: Structure and Form

Three Musical Tuning Systems

Scale Degree	Solfege	Interval	Pythagorean	Just Intonation	Equal Temperament
1	Do	unison (Uni.)	1	1	1
2	Re	major second (M2)	$\frac{9}{8} = 1.125$	$\frac{9}{8} = 1.125$	$2^{2/12} \approx 1.1224620$
3	Mi	major third (M3)	$\frac{81}{64} = 1.265625$	$\frac{5}{4} = 1.25$	$2^{4/12} \approx 1.2599211$
4	Fa	perfect fourth (P4)	$\frac{4}{3} = 1.3\bar{3}$	$\frac{4}{3} = 1.3\bar{3}$	$2^{5/12} \approx 1.3348399$
5	Sol	perfect fifth (P5)	$\frac{3}{2} = 1.5$	$\frac{3}{2} = 1.5$	$2^{7/12} \approx 1.4983071$
6	La	major sixth (M6)	$\frac{27}{16} = 1.6875$	$\frac{5}{3} = 1.6\bar{6}$	$2^{9/12} \approx 1.6817928$
7	Ti	major seventh (M7)	$\frac{243}{128} = 1.8984375$	$\frac{15}{8} = 1.875$	$2^{11/12} \approx 1.8877486$
$8 = 1$	Do	octave (Oct.)	2	2	2

Table 1: The ratios or multipliers used to **raise** a note (increase the frequency) by a given musical interval in the three different tuning systems: Pythagorean Tuning, Just Intonation, and Equal Temperament. The decimal values for the first two tuning systems, which use rational numbers, are exact. On the other hand, those given for Equal Temperament, which uses irrational multipliers (except for unison and the octave), are approximations given to seven decimal places. Note that the major third in Equal Temperament is noticeably sharp from that of Just Intonation, while the perfect fifth is very close to the 1.5 ratio used in both the Pythagorean and Just Intonation tuning systems.

Example: To find the note C$\sharp$ above A440 Hz, we need to go up by a major third. Therefore, in the Pythagorean tuning system we multiply 440 by 1.1.265625 to find that C$\sharp$ is 556.875 Hz. Using Just Intonation, we should tune C$\sharp$ to $440 \times 1.25 = 550$ Hz. Finally, in Equal Temperament, C$\sharp$ would be given by $440 \times 1.2599211 = 554.365284$ Hz.

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Scale Degree	Solfege	Interval	Pythagorean	Just Intonation	Equal Temperament
1	Do	unison (Uni.)	0	0	0
2	Re	major second (M2)	203.9	203.9	200
3	Mi	major third (M3)	407.8	386.3	400
4	Fa	perfect fourth (P4)	498.0	498.0	500
5	Sol	perfect fifth (P5)	702.0	702.0	700
6	La	major sixth (M6)	905.9	884.4	900
7	Ti	major seventh (M7)	1109.8	1088.3	1100
$8 = 1$	Do	octave (Oct.)	1200	1200	1200

Table 2: The ratios or multipliers from the previous table given in terms of the number of **cents**. Note that unlike the previous table, this time the numbers (except for unison and the octave) for the first two tuning systems are irrational (given here as one-decimal approximations) while those for Equal Temperament are exact integers. Why does this happen? Also note the large gap of nearly 14 cents in the major third between Just Intonation and Equal Temperament, while the perfect fifth has a gap of less than 2 cents.

The cents unit of measurement was developed by Alexander Ellis around 1875. It is a common unit of measurement when discussing temperament and tuning. It is based on a logarithmic scale (like decibels). The formula for converting a ratio or multiplier r into cents is

$$\text{Number of cents} \;=\; 1200\log_2(r) \;=\; 1200\frac{\ln r}{\ln 2}$$

Example: A half step in Equal Temperament is given by the multiplier $2^{1/12}$. In cents, using the definition of the logarithm, this is

$$1200\log_2(2^{1/12}) \;=\; 1200 \cdot \frac{1}{12} \;=\; 100 \text{ cents.}$$

Since all half steps are equal in Equal Temperament, one can easily obtain any interval in cents just by multiplying the number of half steps in the interval by 100 (see Table 2). One can compute that the Pythagorean Comma is approximately 23.46 cents while the Syntonic Comma is roughly 21.51 cents.