MONT 108N Exam 2 SOLUTIONS
Math/Music: Structure and Form

December 2, 2011 Prof. G. Roberts

1. [18 pts.] **Pythagorean Tuning**

   (a) In Pythagorean Tuning, what number do you multiply the fundamental frequency of a note by to **raise** it up a perfect fifth?

   Answer: 3/2

   (b) In Pythagorean Tuning, what number do you multiply the fundamental frequency of a note by to **lower** it by an octave?

   Answer: 1/2

   (c) Using your answers to parts **(a)** and **(b)**, **mathematically derive** the number used to **raise** the pitch up a whole step in Pythagorean Tuning.

   Answer: Multiplying the frequency by 9/8 will raise the pitch a whole step in Pythagorean Tuning. To see this, go up by two perfect fifths and then down by one octave (eg. C up P5 = G, G up P5 = D', D' down one octave = D). Thus, using the answers to parts **(a)** and **(b)**, we multiply by

$$\frac{3}{2} \cdot \frac{3}{2} \cdot \frac{1}{2} = \frac{9}{8}.$$

   (d) What is the exact value of the Pythagorean comma? What does it represent and where does it occur in the Pythagorean Tuning system?

   Answer: The Pythagorean comma is equal to $\frac{3^{12}}{2^{19}}$. It represents the margin of error that arises within the Pythagorean Tuning system. It occurs as the gap between two half steps and a whole step,

$$\frac{W}{H^2} = \frac{9/8}{(256/243)^2} = \frac{3^2}{2^3} \cdot \frac{3^{10}}{2^{16}} = \frac{3^{12}}{2^{19}},$$

   as well as the distance between 12 perfect fifths and 7 octaves,

$$\frac{(3/2)^{12}}{2^7} = \frac{3^{12}}{2^{19}}.$$

   This causes the circle of fifths to fail to close up, creating a "spiral" of fifths and musical anomalies such as B$\sharp \neq$ C and F$\sharp \neq$ G$\flat$.

2. [24 pts.] **Fill in the blanks:** Work is **required** to receive partial credit.

   (a) Increasing the amplitude of a sound by 20 decibels increases the volume of sound by a factor of 100.

   Answer: Solving the equation $10 \log_{10} b = 20$ for b yields $\log_{10} b = 2$ or $b = 10^2 = 100$.

   (b) The angle (in radians) that corresponds to going around the unit circle 3.5 times is 7π. The cosine of this angle is -1.

   Answer: Going once around the unit circle (360°) corresponds to an angle of 2π radians. Thus, going around 3.5 times corresponds to an angle of $3.5 \cdot 2\pi = 7\pi$ radians. The cosine of this angle is -1 because the x-coordinate at this point is -1.

(c) The expression $\sin^2(\sqrt{2}\,\pi\theta) + \cos^2(\sqrt{2}\,\pi\theta)$ simplifies to $\underline{1}$.

Answer: The expression simplifies to one because $\cos^2(A) + \sin^2(A) = 1$ for any angle A.

(d) A string of length 24 cm and a string of length 32 cm are plucked simultaneously. The musical interval created between the resulting notes is a perfect fourth.

Answer: The ratio 24/32 simplifies to 3/4 which is the ratio in the Pythagorean scale (string lengths) for a perfect fourth.

(e) In the overtone series, the **musical interval** between the 4th and 5th notes, that is, between $4f$ and $5f$, is a major third.

Answer: This follows because the ratio of $5f$ to $4f$ is 5/4 which is a major third in Just Intonation. Another way to see this is to write out the first 5 notes in the overtone series (starting on any note) and determine the musical interval between the 4th and 5th notes on the staff.

(f) In the Equal Temperament tuning system, the number of cents in a perfect fifth is 700.

Answer: Since there are seven half steps in a perfect fifth and each half step is equivalent to 100 cents, we obtain 700 cents in a perfect fifth (cents is a linear measurement). Alternatively, we have that

$$1200 \log_2(2^{7/12}) \;=\; 1200 \cdot \frac{7}{12} \;=\; 700.$$

3. [18 pts.] **Fun with Trigonometry**

(a) Using the formula

$$\sin(A + B) \;=\; \sin A \cos B + \cos A \sin B,$$

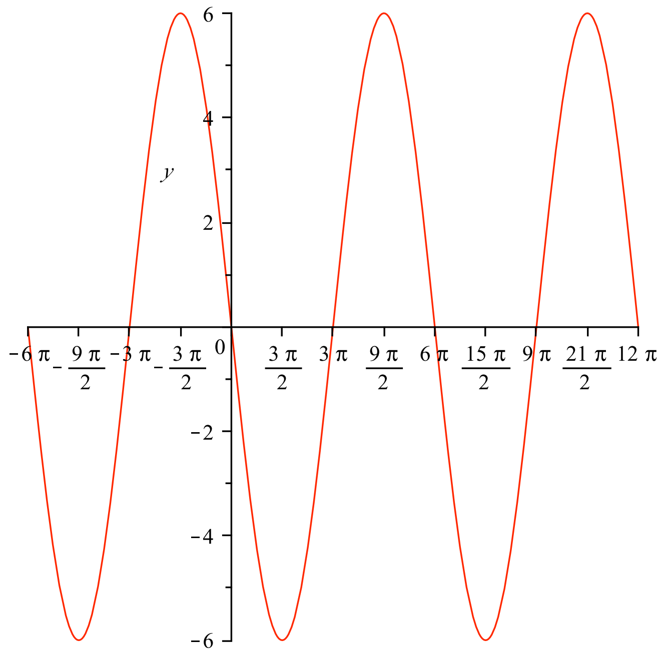
find and simplify $\sin(t + \frac{\pi}{2})$.

Answer:
Let $A = t$ and $B = \frac{\pi}{2}$ and use the formula above. Then

$$\begin{aligned} \sin(t + \frac{\pi}{2}) &= \sin t \cdot \cos\frac{\pi}{2} + \cos t \cdot \sin\frac{\pi}{2} \\ &= \sin t \cdot 0 + \cos t \cdot 1 \\ &= \cos t\,. \end{aligned}$$

(b) Graph the function $y = 6\sin(\frac{1}{3}(t - 3\pi))$. Draw two full periods and label tick marks on the horizontal and vertical axes carefully.

Answer: The amplitude of the graph is 6 while the period is $\frac{2\pi}{\frac{1}{3}} = 6\pi$. The phase shift is 3π so the graph is the usual sine curve shifted to the right by 3π units.



(c) Two notes are played together on two tuning forks, resulting in a sound wave of the form

$$y = 10\sin(250\pi t) + 10\sin(254\pi t)\,.$$

What is the frequency of the "note" we actually hear and how many beats per second are there? Explain.

Answer: The period of the first note is $(2\pi)/(250\pi) = 1/125$, so the frequency is 125 Hz. Similarly, the frequency of the second note is 127 Hz. Using the "rule of beats," the note we hear is the average of the two frequencies, 126 Hz, and the number of beats per second is the difference of the two frequencies, 2 beats/sec.

4. [24 pts.] **Tuning Systems**

(a) Given that the A just below middle C has a frequency of 220 Hz, find the frequency for F♯ above middle C using Pythagorean Tuning. Round your answer to two decimal places.

Answer: 371.25 Hz. We are looking for the F♯ above middle C. This note is a major sixth above the A below middle C. In Pythagorean Tuning the ratio for a major sixth is 27/16, so the frequency for F♯ is $220 \cdot 27/16 = 371.25$ Hz.

(b) Given that the A just below middle C has a frequency of 220 Hz, find the frequency for F♯ above middle C using Just Intonation. Round your answer to two decimal places.

Answer: 366.67 Hz. We are looking for the F♯ above middle C. This note is a major sixth above the A below middle C. In Just Intonation the ratio for a major sixth is 5/3, so the frequency for F♯ is $220 \cdot 5/3 = 366.66\bar{6}$ Hz.

(c) Give two advantages of using Just Intonation as a tuning system.

Answer: Just Intonation is useful for playing in one key (assuming you have tuned the tonic to that key) and particularly nice when playing the major chord of the tonic (1, 3 and 5 of the scale) as well as the IV (subdominant) and V (dominant) major chords. Because the tuning used in Just Intonation matches the overtone series, these chords, which are each in the ratio $4 : 5 : 6$, will sound particularly harmonious. Just Intonation also uses ratios with small integers (again coming from the overtone series) and these are easier to remember than other tuning systems.

(d) Give two advantages of using Equal Temperament as a tuning system.

Answer: Equal Temperament (ET) solves the issue of the spiral of fifths (closing it up to a genuine circle) since 7 octaves in ET is equivalent to 12 perfect fifths (B$\sharp$ = C). Mathematically, this is $(2^{7/12})^{12} = 2^7$. Two half steps equals one whole step in ET and since the octave is equally divided into 12 pieces, it is now possible to switch keys easily using ET. This tuning system is more straight-forward to use than the others because it is solely derived in terms of half steps, where a one half-step rise in pitch is obtained by multiplying the frequency by $2^{1/12}$.

5. [16 pts.] **Irrational and Rational Numbers**

(a) Using Equal Temperament, what number do you multiply the fundamental frequency of a note by to raise it up a minor third? Give the **exact** value (not a decimal). Be sure to simplify your answer.

Answer: $2^{3/12}$ since a minor third is 3 half-steps and one half-step in Equal Temperament is obtained by multiplying by $2^{1/12}$. This number simplifies to $2^{1/4}$.

(b) Prove that your answer to part **(a)** is an irrational number.

Answer: Suppose by contradiction that $2^{1/4}$ is a rational number. Then we can write $2^{1/4} = p/q$ for some positive integers p and q. Raising both sides to the 4th power gives $2 = p^4/q^4$. Cross-multiplying gives

$$2 \cdot q^4 \;=\; p^4.$$

Note that raising an integer to the fourth power will multiply all exponents in the prime factorization of that integer by four, making each exponent an even number. Thus, there are an odd number of 2's (1 plus even or 0 = odd) in the prime factorization of the left-hand side and an even or 0 number of 2's in the prime factorization on the right-hand side. But this contradicts the Fundamental Theorem of Arithmetic which states that the prime factorization of a positive integer is unique. Therefore, our original assumption of rationality has produced a contradiction and $2^{1/4}$ must be an irrational number.

(c) Prove that the **product** of any two rational numbers is also a rational number.

Answer: Suppose that p and q are two rational numbers. Then we can write $p = a/b$ and $q = c/d$ where a, b, c, d are each integers, by the definition of a rational number. Next, we have

$$p \cdot q \;=\; \frac{a}{b} \cdot \frac{c}{d} \;=\; \frac{ac}{bd} \;=\; \frac{m}{n}$$

for some integers m and n. This follows because the product of two integers equals an integer. Thus, $p \cdot q$ can be written as the ratio of two integers and is therefore rational. This completes the proof.