# THE TWO MOST ORIGINAL CREATIONS OF THE HUMAN SPIRIT

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"The science of Pure Mathematics, in its modern developments, may claim to be the most original creation of the human spirit. Another claimant for this position is music."

A. N. Whitehead, *Science and the Modern World.*

**1. Introduction.** In the quotation given above a great Anglo-American philosopher [1] characterized two distinct fields of human interest, one a science, the other an art. The arts and the sciences, however, are not mutually exclusive. Art has often borrowed from science in its attempts to solve its problems and to perfect its achievements. Science in its higher forms has many of the attributes of an art. Vivid aesthetic feelings are not at all foreign in the work of the scientist. The late Professor George Birkhoff, in fact, wrote as follows:

"A system of laws may be beautiful, or a mathematical proof may be elegant, although no auditory or visual experience is directly involved in either case. It would seem indeed that all feeling of desirability which is more than mere appetite has some claim to be regarded as aesthetic feeling." [2]

Serge Koussevitzky, noted conductor, has stated also that "there exists a profound unity between science and art." [3]

It is not, however, the purpose of this paper, to discuss the relationships between the sciences and the arts, but rather to enumerate some of the lesser known attributes which music and mathematics have in common. There is no attempt to establish a thesis.

**2. Number and Pitch.** The study of mathematics usually begins with the natural numbers or positive integers. Their symbolic representation has been effectively accomplished by means of a radix or scale of ten, the principle of place-value where the position of a digit indicates the power of ten to be multiplied by it, and a zero. The concept of number is most basic in mathematics. We cannot directly sense number. A cardinal number, such as five, is an abstraction which comes to us from many concrete instances each of which possesses other attributes not even remotely connected with the one upon which our interest is fixed. Such widely differing groups as the fingers of the hand, the sides of the pentagon, the arms of a starfish, and the Dionne quintuplets, are all instances of "fiveness," the property which enables each group to be matched or placed into one-to-one correspondence with the other. The establishment of such equivalence requires no knowledge of mathematics, only good eyesight. With these facts in mind we may state a definition familiar to mathematicians. *The (cardinal) number of a group of objects is the invariant property of the group and all other groups which can be matched with it.*

The positive integers constitute, however, but a small portion of the numbers of mathematics. The former mark off natural intervals in the continuum of real numbers. The difference between two small groups of objects is readily sensed; man finds no difficulty in distinguishing *visually,* at once, between *three* and *four* objects, but the distinction between, say, thirty-two and thirty-three objects calls for something more than good vision.

In music, study begins with notes or tones. In western music their symbolic representation is accomplished by means of a scale of seven, a principle of position, and the rest, which denotes cessation of tone. There is something permanent and unchangeable about a given note. You may sing it, the violin string may emit it, the clarinet may sound it, and the trumpet may fill the room with it. The quality or timbre, the loudness or intensity, and the duration of one sound may be markedly different from another; yet among these differences of sound there remains one unchanging attribute, its pitch. This is the same for a single such note or any combination of them. The pitch of a note may then be defined as *the invariant property of the note and all other notes which may be matched with it.* Notes which can be matched are said to be in unison. Pitch, also, is an abstraction, derived from many auditory experiences. The establishment of pitch equivalence does not require a knowledge of music, only a keen ear.

The notes of the diatonic scale mark off convenient intervals in a continuum of pitches. Within a given range, the interval between two tones of the scale is, in general, readily sensed, but outside of such a range the human ear may fail to distinguish between or even to hear two differing tones. As a matter of fact, "tones" removed from the range of audibility cease to be such. As psychological entities they disappear and may be identified only as vibrations in a physical medium.

Invariance of pitch is an important musical property and the ability of a musician not playing a keyed instrument to maintain this property for a given note is a necessary, but not a sufficient condition for his artistry. This recalls the story of the distracted singing teacher who, after accompanying his none-too-apt pupil, sprang suddenly from the piano, thrust his fingers wildly through his hair, and shouted, "I play the white notes, and I play the black notes, but you sing in the cracks."

**3. Symbols.** Mathematics is characterized by an extensive use of symbols. They are indispensable tools in the work; they constitute the principal vehicle for the precise expression of ideas; without them modern mathematics would be non-existent. The most important mathematical symbols are, with few exceptions, in universal use among the civilized countries of the world.

Music also is distinguished by a universal symbolism. The creation of anything but the simplest musical composition or the transmission of significant musical ideas is difficult if not impossible without the symbols of music.

Incidentally it may be remarked that the page of a musical score and the page of a book in calculus are equally unintelligible to the uninitiated.

There are few fields of activity outside of mathematics (including logic) and music which have developed so extensively their own symbolic language. Chemistry and phonetics are nearest in this respect.

In both music and mathematics preliminary training involves the acquiring of technique. Mathematics demands such facile manipulation of symbols that the detailed operations become mechanical. We are encouraged to eliminate the necessity for elementary thinking as much as possible, once the fundamental logic is made plain. This clears the way for more complicated processes of reasoning.

"It is a profoundly erroneous truism, repeated by all copy-books and by eminent people when they are making speeches, that we should cultivate the habit of thinking of what we are doing. The precise opposite is the case. Civilization advances by extending the number of important operations which we can perform without thinking about them." [4]

In music also, the preliminary training involves a learning of technique. The aim here is to be able to read, or to write, or to translate into the appropriate physical actions, notes and combinations of them with such mechanical perfection that the mind is free for the creation and the interpretation of more profound musical ideas.

**4. Logical Structure.** The framework of a mathematical science is well known. We select a class of objects and a set of relations concerning them. Some of these relations are assumed and others are deduced. In other words, from our axioms and postulates we deduce theorems embracing important properties of the objects involved.

Music likewise has its logical structure. The class of objects consists of such musical elements as tones, intervals, progressions, and rests, and various relations among these elements. In fact, the structure of music has been formally described as a set of postulates according to the customary procedure of mathematical logic. [5]

In mathematics a development is carried forward according to the axioms or postulates. If these are obeyed the results are correct, in the mathematical sense, although they may not be interesting or useful. Mere obedience to law does not create an original piece of mathematical work. This requires technical skill, imagination, and usually a definite objective.

Music also has its axioms or laws. These may be as simple as the most obvious things in elementary mathematics – the whole equals the sum of all its parts – if we are counting beats in a measure; they may be less obvious to the layman, such as the canons of harmony or the structural laws of a classical symphony. Here again we may follow the laws of music scrupulously without ever creating a worth-while bit of original music. Technical skill, imagination, the fortunate mood, and usually a definite objective are requisites for the creation of a composition which not only exhibits obedience to musical laws but expresses significant ideas also. Occasionally the musician becomes bold and violates the traditional musical axioms so that the resulting effects may at first sound strange or unpleasant. These may become as useful, provoking, and enjoyable, as a non-Euclidean geometry or a non-Aristotelian logic. In such manner did Wagner, Debussy, Stravinsky, and others extend the bounds of musical thought. In mathematics as well as in music one may have to become accustomed to novel developments before one learns to like them.

Benjamin Peirce defined mathematics as "the science which draws necessary conclusions." The operations from hypothesis to theorem proceed in logical order without logical hesitation or error. When the series of deductive operations flows swiftly and naturally to its inevitable conclusion, the mathematical structure gives a sense of satisfaction, beauty, and completeness. Sullivan characterizes the opening theme of Beethoven's Fifth Symphony as one which "immediately, in its ominous and arresting quality, throws the mind into a certain state of expectance, a state where a large number of happenings belonging to a certain class, can logically follow." [6] The same is true of the opening phrase of the prelude to *Tristan and Isolde*, or of any really great enduring masterpiece.

An interesting departure from the usual logical structure of a musical composition occurs in the Symphonic Variations, "Istar" by Vincent d'Indy. Instead of the initial announcement of the musical theme with its subsequent variations, "the seven variations proceed from the point of complex ornamentation to the final stage of bare thematic simplicity." Philip Hale, the eminent Boston musical critic related the following anecdote in the Boston Symphony Bulletin of April 23, 1937.

"M. Lambinet, a professor at a Bordeaux public school, chose in 1905 the text 'Pro Musica' for his prize-day speech. He told the boys that the first thing the study of music would teach them would be logic. In symphonic development logic plays as great a part as sentiment. The theme is a species of axiom, full of musical truth, whence proceed deductions. The musician deals with sounds as the geometrician with lines and the dialectitian with arguments. The master went on to remark: 'A great modern composer, M. Vincent d'Indy, has reversed the customary process in his symphonic poem "Istar." He by degrees unfolds from initial complexity the simple idea which was wrapped up therein and appears only at the close, like Isis unveiled, like a scientific law discovered and formulated.' The speaker found this happy definition for such a musical work – 'an inductive symphony.' "

**5. Meaning.** A mathematical formula represents a peculiarly succinct and accurate representation of meaning which cannot be duplicated by any other means. It is concerned with the phenomenon of variability; it involves the function concept. "A mathematical formula can never tell us what a thing is, but only how it behaves." [7]

How true this is of music! A theme of great music compresses into a small interval of space or time, inimitably and accurately, a remarkable wealth of meaning. Music is not fundamentally concerned with the description of static physical objects, but with the impressions they leave under varying aspects. Debussy's "La Mer" is a fine example of this type of description. Music's interest is often not in the physical man but in his changing moods, in his emotions. One of the sources of the greatness of "Die Walküre" is Wagner's genius for portraying vividly the conflicting aspects of Wotan's nature—as god and as man.

The meaning of musical motive grows with study. It is usually exploited or developed and from it are derived new figures of musical expression. A good theme demands more than the casual hearing before its deep significance is completely appreciated. It is often worked up from an entirely insignificant motive as in Beethoven's Fifth Symphony or in Mozart's G minor symphony. In mathematics a basic formula or equation may have implications which can be understood only after much study. It may appear to be almost trivial as in the case of $a+b = b+a$ or it may be less obvious and more elegant as in the case of Laplace's equation,

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 .$$

Music consists of abstractions, and at its best gives expression to concepts which represent the most universal features of life. Beethoven's music expresses powerfully the great aspirations, struggles, joys, and tragedies of human existence. The Eroica symphony may have been composed with Napoleon in mind but it portrays far more than the career of a single man. It is a portrayal of the heroic in man and as such is universal in its application. It is well known that a musical passage or composition may produce different responses among people. The possibility of varying interpretation constitutes one of the sources of music's uniqueness and a reason for its power. It is an evidence of its universality. Herein lies a fundamental difference between music and painting or sculpture. The effect of a musical episode is due to its wide potential emotional applicability; the effect of a painting or piece of sculpture is due to its concreteness. Attempts at abstract representations by painters have not been generally successful; attempts at stark realism in music have likewise failed. Music in its most abstract form, as for example, Bach's or Mozart's, often defies application to the concrete. It seems to be above mundane things, in the realm of pure spirit.

So it is with mathematics. Our conclusions are always abstract, and universal in their application, although they may have originated from a special problem. The possibilities of interpretation and application of a given theorem or formula are unlimited. Poincaré is reported to have said that even the same mathematical theorem has not the same meaning for two different mathematicians. What differing reactions may ensue when Laplace's equation is set up before an audience of mathematicians! What differing degrees of abstractness are suggested by the two equations previously written!

**6. The Creative Process.** "It is worth noting ... that it is only in mathematics and music that we have the creative infant prodigy; ... the boy mathematician or musician, unlike other artists, is not utilizing a store of impressions, emotional or other, drawn from experience or learning; he is utilizing inner resources. ..." [8]

Statements of this type have led many to believe that mathematical talent and musical talent have more than an accidental relation. Some feel that mathematicians are more naturally drawn to music than musicians are to mathematics. As far as the writer has been able to ascertain, no serious investigations on the relation between the two talents have been published. A brief study of exceptionally gifted children yields no testimony that the child prodigy in music has more than the average mathematical sense, or that the child prodigy in mathematics has exceptional ability in music.

In a recent article, *Mind and Music,* [9] the inimitable English music critic, Ernest Newman, discusses the role that the subconscious mind might play in the creative processes of music. Hampered by a dearth of reliable testimony on this subject, he attempts, nevertheless, to estimate this role. Berlioz and Wagner had written of their creative experiences without attempting any self-analysis. So also had Mozart although Newman does not refer to him. Newman feels that the Memoirs of Hector Berlioz are not too reliable in this respect. Wagner's letters, however, seem to indicate that many of his musical ideas were the result of an upsurge from the unconscious depths of his mind of ideas long hidden but suddenly crystallizing. The activity of his conscious mind was often displaced by the upward thrust of these latent creative forces.

The interpretation thus suggested is strikingly similar in many respects to that described by Jacques Hadamard in his *Essay on the Psychology of Invention in the Mathematical Field.* [10] This noted mathematician draws on the related experiences of Poincaré, Helmholtz, Gauss, and others to discuss the origin of the inspiration or sudden insight that contributes to, completes, or initiates an original work. The role of thoughts that lie vague and undiscernible in the subconscious, only to become, of a sudden, clear and discernible after a period of unsuspected incubation, is described in undogmatic terms. One cannot affirm, of course, that these opinions concerning the creative process are confined solely to music and mathematics, but it is interesting that they are voiced by two eminent scholars, one from each field.

The greatest works of music are distinguished by their intellectual content as well as by their emotional appeal. The sacred music of Bach, the symphonies of Beethoven, or the operas of Wagner, offer subjects for analysis and discussion, as well as opportunities for emotional experience. Each composer had ideas to "work out," ideas to be developed and clarified by the forms and artifices of music, the object being to make their full significance felt by the appreciative listener.

Mathematical creativity involves very much the same general development. Concepts must be clarified, operations carried out, latent meanings revealed. If these are significant and logically developed the result has a unity and a sense of completeness which brings intellectual and aesthetic satisfaction to both author and reader.

**7. Aesthetic Considerations.** To many, mathematics seems to be a forbidding subject. Its form seems to be more like that of a skeleton than that of a living, breathing, human body. This idea, is, of course, derived from its abstract character and from the demands which it makes for sharply defined concepts, terse methods of expression, and precise rules of operation. In a sense, mathematics lacks richness, if by richness we mean the presence of those impurities which impart savor and color. These impurities may be in the nature of concrete examples, illustrations from, or applications to fields other than mathematics. They may represent departures from the normal abstract logical development, and may make no contributions whatsoever to the formal structure which constitutes mathematics. But if we subtract from the richness of mathematics, we add also to its purity, for in mathematics the structure or form is more important than its applications. We may apply the mathematics to many problems associated with human existence, but these applications are not essential parts of pure mathematics, they lie apart from it.

"In music the flavor of beauty is purest, but because it is purest it is also least rich. ... A melody is a pure form. Its content is its form and its form is its content. A change in one means a change in other. We can, of course, force an external content upon it, read into it stories or pictures. But when we do so we know that they are extraneous and not inherent in the music."[11]

In a different sense mathematics is over-rich for its fields are unlimited in extent and fertility.

"But no one can traverse the realm of the multiple fields of modern Mathematics and not realize that it deals with a world of its own creation, in which there are strangely beautiful flowers, unlike anything to be found in the world of external entities, intricate structures with a life of their own, different from anything in the realm of natural science, even new and fascinating laws of logic, methods of drawing conclusions more powerful than those we depend upon, and ideal categories very widely different from those we cherish most."[12]

One needs here but to change a few words in order to describe the unique and lovely creations of music. The melodies and harmonies of music are its own inventions. They are often mysteriously beautiful, incapable of description by other means and without counterpart elsewhere in the world about us. A musical composition may be of the utmost simplicity or of the most intricate character, yet it may "well-nigh express the inexpressible." It is exactly this ability to convey the "inexpressible" ideas that give mathematics and music much in common. The mathematics student who seeks always a meaning or picture of each new proposition often fails to appreciate the power of that which defies representation.

**8. Conclusion.** There is much of interest to those who love both music and mathematics, and much has been written by mathematicians on the bearings of one field on the other. Archibald has written delightfully of some of their human aspects as well as the scientific. Birkhoff has attempted the evaluation of musical aesthetics by quantitative methods. Miller and others have brought the instruments of physics to bear upon the problems of musical tone and acoustics.

Success in music and in mathematics also depend upon very much the same things — fine technical equipment, unerring precision, and abundant imagination, a keen sense of values, and, above all, a love for truth and beauty.

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## Maybe

*Within your lifetime will, perhaps,*
*As souvenirs from distant suns*
*Be carried back to earth some maps*
*Of planets and you'll find that one's*
*So hard to color that you've got*
*To use five crayons. Maybe, not.*

Marlow Sholander

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