Math/Music: Structure and Form

Strähle's Guitar Construction

Strähle's placement of the frets on the guitar corresponds to the linear fractional transformation

$$y = \frac{17 - 5x}{17 + 7x}.$$

Here, we break the base of the isosceles triangle (the x-axis) into 12 equal parts and label the rightmost point $x = 0$ (the point R in Figure 5 in the text) and the leftmost point $x = 1$ (the point Q in Figure 5). The neck of the guitar containing the frets is the y-axis with the point R labeled $y = 1$ and the point M (where the string is fastened to the base of the guitar) is labeled $y = 0$. This means that $y = 1/2$ is at the point P, the midpoint of the string. Consequently, placing your finger at P and plucking the string yields a note an octave higher than the fundamental note of the full string. Thus, the value of y is precisely the ratio between the lengths of the original string and the plucked string when placing a finger down on the fret at point y. Moreover, since we have subdivided the base of the triangle into 12 equal parts, with $(x = 0, y = 1)$ corresponding to the fundamental note and $(x = 1, y = 1/2)$ corresponding to an octave higher, the value of y for each x-value of the form $x = n/12$, gives Strähle's frequency ratio for moving up n half steps in pitch.

For example, $y = 1$ (point R), means pluck the original string, so the frequency ratio is 1/1 or $y = 1$. In class, we showed that $y = 29/41$ is the value where the altitude of the isosceles triangle intercepts the guitar string. Therefore, setting $n = 6$, we have that $x = 1/2$ maps to $y = 29/41$ (check it in the formula above) and thus 6 half steps (the tritone) is a ratio of 29/41. This is an excellent approximation to the ratio obtained in Equal Temperament:

$$6 \text{ half steps} = 2^{6/12} = 2^{1/2} = \sqrt{2}.$$

However, since we are comparing lengths of strings and this is the *inverse* of frequency, we really want to compare with $1/\sqrt{2}$. Therefore, if Strähle's method is to make any sense at all, we would need 29/41 to approximate $1/\sqrt{2}$. Inverting each fraction, this is equivalent to

$$\frac{41}{29} \approx 1.4138 \quad \text{approximates} \quad \sqrt{2} \approx 1.4142,$$

an excellent approximation!

We have seen the number 41/29 before. It is the fifth convergent p_4/q_4 in the continued fraction expansion of $\sqrt{2}$ (see the top of page 4 on the *Continued Fractions* handout). Recall that the convergents obtained from the continued fraction expansion of an irrational number are the *best* approximating rational numbers with small denominators. The fact that Strähle's method just happens to use a convergent from the continued fraction expansion for $\sqrt{2}$ is one of the primary reasons his method is so effective.

Table 1 compares the ratios for Strähle's method with those of Equal Temperament. Note that the maximum total error is only 0.152%. Also note the rather large values in the fractions. Table 2 shows a comparison of the inverse of these values (frequency ratios) in terms of cents, with Just Intonation included. Remarkably, the discrepancy between Strähle's method and Equal Temperament is never more than 3 cents.

Note	Interval	Strähle	Equal Temp.	% Error
C	unison	$\frac{1}{1}$	1	0
C$\sharp$	minor second	$\frac{199}{211} \approx 0.9431$	$2^{-1/12} \approx 0.9439$	-0.079
D	major second	$\frac{97}{109} \approx 0.8899$	$2^{-1/6} \approx 0.8909$	-0.111
D$\sharp$	minor third	$\frac{21}{25} = 0.84$	$2^{-1/4} \approx 0.8409$	-0.107
E	major third	$\frac{23}{29} \approx 0.7931$	$2^{-1/3} \approx 0.7937$	-0.075
F	perfect fourth	$\frac{179}{239} \approx 0.7490$	$2^{-5/12} \approx 0.7492$	-0.027
F$\sharp$	tritone	$\frac{29}{41} \approx 0.7073$	$2^{-1/2} \approx 0.7071$	0.030
G	perfect fifth	$\frac{169}{253} \approx 0.6680$	$2^{-7/12} \approx 0.6674$	0.085
G$\sharp$	minor sixth	$\frac{41}{65} \approx 0.6308$	$2^{-2/3} \approx 0.6300$	0.128
A	major sixth	$\frac{53}{89} \approx 0.5955$	$2^{-3/4} \approx 0.5946$	0.152
A$\sharp$	minor seventh	$\frac{77}{137} \approx 0.5620$	$2^{-5/6} \approx 0.5612$	0.145
B	major seventh	$\frac{149}{281} \approx 0.5302$	$2^{-11/12} \approx 0.5297$	0.098
C	octave	$\frac{1}{2}$	$\frac{1}{2}$	0

Table 1: Comparing the ratios of Strähle's method and Equal Temperament. Note that the largest total error is only 0.152%.

Note	Interval	Strähle	Equal Temp.	Just Int.
C	unison	0	0	0
C♯	minor second	101.37	100	111.7
D	major second	201.9	200	203.9
D♯	minor third	301.8	300	315.6
E	major third	401.3	400	386.3
F	perfect fourth	500.5	500	498.0
F♯	tritone	599.5	600	590.2
G	perfect fifth	698.5	700	702.0
G♯	minor sixth	797.8	800	813.7
A	major sixth	897.4	900	884.4
A♯	minor seventh	997.5	1000	996.1
B	major seventh	1098.3	1100	1088.3
C	octave	1200	1200	1200

Table 2: Comparing the frequency ratios (multipliers) of Strähle's method, Equal Temperament and Just Intonation in terms of cents. Note that Strähle's method is never more than 3 cents away from Equal Temperament.