Math/Music: Structure and Form   Fall 2010

Continued Fractions

1 Playing around with π

The famous mathematical constant π is irrational, with a non-repeating, non-terminating decimal expansion. To 40 decimal places, π is

$$\pi = 3.1415926535897932384626433832795028841972\ldots.$$

Since π is irrational, it can **not** be written as the ratio of two integers. However, we could try and *approximate* π using rational numbers. This suggests the question: "How does one obtain good approximations to irrational numbers, particularly with smaller numerators and denominators?" This last emphasis harkens back to our discussion of the Pythagoreans and to the Just Intonation tuning system with its use of ratios with small numbered fractions.

Begin by writing π as

$$\pi = 3 + 0.1415926535897\ldots.$$

One approximation for π is thus three (this actually occurs in the Bible), but that's not all that impressive. We could continue by approximating 0.14 as $14/100 = 7/50$ in order to obtain the first two decimal places of π. However, here's a more clever approach. Instead of approximating $0.14\ldots$ as $7/50$, consider inverting it twice and writing

$$0.1415926535897\ldots = \frac{1}{\dfrac{1}{0.1415926535897\ldots}} = \frac{1}{7.062513305931\ldots}.$$

Ignoring the decimals after $7.06\ldots$, this suggests the approximation $3 + 1/7 = 22/7$, a famous approximation to π you probably learned about in high school. Like the fraction $157/50 = 3+7/50$, this gives π to two decimal places, but notice that $22/7$ has a denominator about 7 times *smaller* than $157/50$. This is nice.

But why stop there. We can invert the new "remainder" decimal $0.062513305931\ldots$ twice as well, obtaining

$$0.062513305931\ldots = \frac{1}{\dfrac{1}{0.062513305931\ldots}} = \frac{1}{15.99659441\ldots}.$$

This last number in the denominator is very close to 16. If we approximate at this point, we find

$$\pi \approx 3 + \frac{1}{7 + \frac{1}{16}} = 3 + \frac{16}{113} = \frac{355}{113} = 3.14159292\ldots.$$

Remarkably, the approximation $355/113$ agrees with π to six decimal places! If we tried to get this approximation by truncating π to 6 decimal places, we would have a reduced fraction with denominator $125,000$. That's roughly 1100 times larger than 113.

If we continue the process of inverting the remainder decimal two more times, we obtain the following multi-layered fraction,

$$\pi \approx 3+\cfrac{1}{7+\cfrac{1}{15+\cfrac{1}{1+\frac{1}{293}}}} = \frac{104348}{33215} = 3.14159265392142\ldots \tag{1}$$

which approximates π accurately to 9 decimal places!

These crazy, multistory fractions are called **continued fractions** and are an important topic in the subject of number theory, although they find there way into all sorts of applications in other fields.

2 Continued Fractions

Definition 2.1 *Given a real number α, the continued fraction expansion of α is*

$$\alpha = a_0+\cfrac{1}{a_1+\cfrac{1}{a_2+\cfrac{1}{a_3+\cfrac{1}{\ddots}}}} = [a_0; a_1, a_2, a_3, \ldots]$$

where each a_i (except possibly a_0) is a positive integer.

The notation $\alpha = [a_0; a_1, a_2, a_3, \ldots]$ is much easier to use than writing the multistory fraction. The first integer a_0 is just the number to the left of the decimal point (eg. $a_0 = 3$ for $\alpha = \pi$). Note the semi-colon after the a_0. The succeeding list of integers are the denominators of each fraction since we know the numerators will always be 1's. For example, from equation (1), we have that

$$\pi = [3; 7, 15, 1, 292, \ldots].$$

The reason for the 292 and not 293 is that the final approximation we did for π had a $292.63459\ldots$, so although we rounded this to 293 to obtain the approximation 104348/33215, the actual integer *before* rounding is 292.

Note that if the number we are approximating is irrational, then this process of inverting the remainder decimal twice will continue forever; otherwise, the approximation would be exact and we would have a rational number.

Theorem 2.2 *The continued fraction expansion of a real number is finite if and only if that number is rational. In other words, the continued fraction expansion of an irrational number is infinite.*

To see how this process works on a rational number, let's try an example.

Example 2.3 *Compute the continued fraction expansion for the rational number $\alpha = 37/13$.*

Answer: We start by dividing 37 by 13 to obtain 2 with a remainder of 11. Thus,

$$\frac{37}{13} = 2+\frac{11}{13} = 2+\frac{1}{13/11}.$$

Note that the final fraction in the denominator is bigger than one. This will always be the case since we are inverting a number less than one (the remainder). Continuing, we divide 13 by 11 to obtain 1 with a remainder of 2. Now we have

$$\frac{37}{13} = 2 + \frac{11}{13} = 2 + \frac{1}{13/11} = 2 + \cfrac{1}{1 + \cfrac{2}{11}} = 2 + \cfrac{1}{1 + \cfrac{1}{11/2}}$$

Next, we divide 11 by 2 to obtain 5 with a remainder of 1. This gives

$$\frac{37}{13} = 2 + \frac{11}{13} = 2 + \frac{1}{13/11} = 2 + \cfrac{1}{1 + \cfrac{2}{11}} = 2 + \cfrac{1}{1 + \cfrac{1}{11/2}} = 2 + \cfrac{1}{1 + \cfrac{1}{5 + \cfrac{1}{2}}}$$

The process terminates at this point because we are left with the integer 2 in the denominator rather than a fraction. There is nothing left to divide when the remainder is one. The last fraction of $11/2$ is simply $5 + 1/2$ which already is in the required form because of the one in the numerator. Note that all the previous steps did **not** have a one for a remainder.

Key Fact: When writing the finite continued fraction expansion of a rational number, the process terminates precisely when a remainder of one occurs.

Thus, we have shown that the continued fraction expansion for the rational number $37/13$ is

$$\frac{37}{13} = [2; 1, 5, 2] .$$

2.1 Periodic Expansions

Sometimes, instead of terminating, the continued fraction expansion will repeat a certain pattern forever. For example,

$$\sqrt{8} = [2; 1, 4, 1, 4, 1, 4, 1, 4, \ldots]$$

has a repeating pattern of **period** 2 since it ends with $1, 4, 1, 4, \ldots$. When this occurs, we write

$$\sqrt{8} = [2; \overline{1, 4}].$$

Another example is

$$\sqrt{7} = [2; 1, 1, 1, 4, 1, 1, 1, 4, 1, 1, 1, 4, \ldots] = [2; \overline{1, 1, 1, 4}].$$

Example 2.4 *Find the irrational number that corresponds to the continued fraction expansion*

$$\alpha = [1; 2, 2, 2, \ldots] = [1; \overline{2}].$$

Answer: One way to approach the problem is to approximate α by terminating the expansion at different locations. These approximations are called **convergents**. In general, the

$$n^{\text{th}} \text{ convergent to } \alpha = \frac{p_n}{q_n} = [a_0; a_1, a_2, a_3, \ldots, a_n] .$$

The larger n is (the further out in the expansion we go), the better the approximation to α becomes.

For $\alpha = [1; 2, 2, 2, \ldots]$, the first five convergents are given below:

$$\begin{aligned}
\frac{p_0}{q_0} &= 1 \qquad \text{(this is just } a_0\text{)} \\
\frac{p_1}{q_1} &= 1 + \frac{1}{2} = \frac{3}{2} = 1.5 \\
\frac{p_2}{q_2} &= 1 + \cfrac{1}{2 + \cfrac{1}{2}} = \frac{7}{5} = 1.4 \\
\frac{p_3}{q_3} &= 1 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2}}} = \frac{17}{12} = 1.41\bar{6} \\
\frac{p_4}{q_4} &= 1 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2}}}} = \frac{41}{29} \approx 1.4138
\end{aligned}$$

Do you recognize the last approximation of 1.414? This suggests that $\alpha = \sqrt{2}$ which turns out to be true! Notice that not only are the convergents getting closer to $\sqrt{2}$ but they also oscillate about $\sqrt{2}$ — below, above, below, above, etc. This will always be the case, no matter the value of α.

There is a clever trick to see that $\alpha = \sqrt{2}$. Since the continued fraction pattern is periodic, the expression for α actually *reappears* inside its own expansion. We have

$$\alpha = 1 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{\ddots}}}} = 1 + \cfrac{1}{1 + 1 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{\ddots}}}} = 1 + \frac{1}{1 + \alpha}.$$

This last expression may seem bizarre, but if we know that the infinite continued fraction converges to a real number (this requires a proof), then the continued fraction expansion in the denominator really is equal to the original continued fraction expansion for α. The main concept here (which we'll study in more detail next semester) is called **self-similarity** and is a key ingredient in the fascinating field of **Chaos Theory**.

Thus, we have that

$$\begin{aligned}
\alpha = 1 + \frac{1}{1+\alpha} \quad &\Longrightarrow \quad \alpha - 1 = \frac{1}{1+\alpha} \\
&\Longrightarrow \quad \alpha^2 - 1 = 1 \qquad \text{(by cross-multiplying)} \\
&\Longrightarrow \quad \alpha^2 = 2 \\
&\Longrightarrow \quad \alpha = \sqrt{2}.
\end{aligned}$$

Note that we choose the positive square root because it is clear from the continued fraction expansion that $\alpha > 0$.

Key Fact: The continued fraction expansion of an irrational number is **periodic** if and only if α is of the form $\alpha = r + s\sqrt{n}$, where r, s are rational numbers and n is a positive integer not equal to a perfect square. In this case, periodic means that the continued fraction expansion ends with a periodic, repeating sequence of positive integers.

2.2 Convergents and Their Accuracy

Instead of having to compute the value of the convergents from scratch each time, there are some nice **recursion formulas** available that give the next convergent in terms of the previous entries. The formulas are:

$$\begin{aligned} p_0 &= a_0 \\ p_1 &= a_1 a_0 + 1 \\ p_n &= a_n\, p_{n-1} + p_{n-2} \quad \text{if } n \geq 2 \end{aligned}$$

and

$$\begin{aligned} q_0 &= 1 \\ q_1 &= a_1 \\ q_n &= a_n\, q_{n-1} + q_{n-2} \quad \text{if } n \geq 2. \end{aligned}$$

For example, if $\alpha = \sqrt{2} = [1; 2, 2, 2, \ldots]$, then $p_0 = 1, p_1 = 2 \cdot 1 + 1 = 3$,

$$\begin{aligned} p_2 &= a_2\, p_1 + p_0 = 2 \cdot 3 + 1 = 7 \\ p_3 &= a_3\, p_2 + p_1 = 2 \cdot 7 + 3 = 17 \\ p_4 &= a_4\, p_3 + p_2 = 2 \cdot 17 + 7 = 41 \end{aligned}$$

A similar set of calculations can be used to find the denominators q_n. Try checking them with $\sqrt{2}$ and comparing with the convergents we obtained above.

One final point concerning the accuracy of the sequence of convergents to the irrational number α they approximate. It turns out that the rational approximations to an irrational number α obtained by using a continued fraction expansion are the *best* possible for a given denominator. Moreover, each convergent is closer to α than the preceding one and the convergents oscillate about α, with the even convergents ($n = 0, 2, 4, \ldots$) lying below α and the odd convergents ($n = 1, 3, 5, \ldots$) lying above. Specifically, we have

Theorem 2.5 *If $\{p_n/q_n\}$ is the sequence of convergents to an irrational number α, then*

$$\frac{p_n}{q_n} - \frac{p_{n-1}}{q_{n-1}} = \frac{(-1)^{n+1}}{q_{n-1}\, q_n}.$$

Moreover, any particular convergent p_n/q_n is closer to α than any other fraction whose denominator is less than q_n.

For example, when approximating $\alpha = \sqrt{2}$, choosing the fifth convergent $41/29$ gives the best possible rational approximation to $\sqrt{2}$ with denominator less than 29. This is precisely why the convergents of a continued fraction expansion are considered the *best* rational approximations.

3 Exercises

1. Compute the continued fraction expansion for the rational number $\alpha = 53/14$. Be sure to show your work. Give your answer in the form $[a_0; a_1, a_2, \ldots]$.

2. Compute the continued fraction expansion for the irrational number $\alpha = \sqrt{6}$. Be sure to show your work. Give your answer in the form $[a_0; a_1, a_2, \ldots]$. *Hint:* Your answer should be periodic.

3. Consider the irrational number α with periodic continued fraction expansion $[1; 1, 1, 1, \ldots]$.

   a) Compute the first seven convergents of α, that is, compute p_n/q_n for $n = 0, 1, \ldots, 6$. What do you notice about the numbers in the numerator and denominator?

   b) Find the exact value of α using the method of self-similarity demonstrated above with $\sqrt{2}$. What is the name of the number α?

4. Recall the importance of the number $\log_2(3/2)$ when attempting to find good approximations to a true or just perfect fifth.

   a) Compute the first seven convergents of $\log_2(3/2)$, that is, compute p_n/q_n for $n = 0, 1, \ldots, 6$. Be sure to show your work.

   b) If you have done part **a)** correctly, you should recognize the fraction obtained for $n = 4$. Explain how this relates to Equal Temperament.

   c) Why would it "work" to divide the octave into 53 equal parts, creating a scale with 53 notes equally spaced by the half step $H = 2^{1/53}$? In this case, what would the value of the frequency multiplier be to raise the pitch a perfect fifth? (This is the number to multiply the tonic frequency by to raise the pitch a P5.) How many cents is this ratio (to three decimal places) and how close is it (in cents, three decimal places) to a just perfect fifth?

   d) Assuming you created a 53 note scale with equally spaced half steps, how many half steps would be in a major third? In this case, how many cents is the ratio to raise the pitch a major third (to three decimal places)? How close is it (in cents, three decimal places) to a just major third? Would you need to raise or lower the pitch in your new scale to obtain a just major third?

References

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[3] E. Weisstein. *Continued Fraction*, MathWorld: A Wolfram Web Resource, `http://mathworld.wolfram.com/ContinuedFraction.html`