MONT 110N-(01,02) Exam 2 SOLUTIONS

Math/Music: Structure and Form

November 22, 2010 Prof. G. Roberts

1. **The Pythagorean Scale**

   (a) In the Pythagorean scale, what ratio is multiplied by the length of a string so that the new string sounds a perfect fifth **higher**? (3 pts.)

   Answer: 2/3. Note that the question concerns the **length** of the string not the frequency. Thus, we multiply by a number less than one.

   (b) In the Pythagorean scale, what ratio is multiplied by the length of a string so that the new string sounds an octave **lower**? (3 pts.)

   Answer: 2

   (c) Using your answers to parts **(a)** and **(b)**, **derive** the ratio that when multiplied by the length of a string, raises the pitch a whole step. (6 pts.)

   Answer: Multiplying the length by 8/9 will raise the pitch a whole step in the Pythagorean scale. To see this, go up by two perfect fifths and then down by one octave (eg. C up P5 = G, G up P5 = D', D' down one octave = D). Thus, using the answers to parts **(a)** and **(b)**, we multiply by

$$\frac{2}{3} \cdot \frac{2}{3} \cdot 2 = \frac{8}{9}.$$

   (d) What is the exact value of the Pythagorean comma? Give two musical problems that arise when using the Pythagorean scale. (6 pts.)

   Answer: The Pythagorean comma is equal to $\frac{3^{12}}{2^{19}}$. Some of the musical problems that arise with the Pythagorean scale are that it is hard to change to other keys, the circle of fifths does not close up creating a spiral of fifths (this is due to the Pythagorean comma) and that two half steps do not equal a whole step ($H^2 \neq W$).

2. **Fun with Trigonometry** (6 pts. each)

   (a) The graph of $\cos t$ is the same as the graph of $\sin t$ but shifted to the left by $\pi/2$. In other words, $\sin(t + \frac{\pi}{2}) = \cos t$. Using the formula

$$\sin(A + B) = \sin A \cos B + \cos A \sin B$$

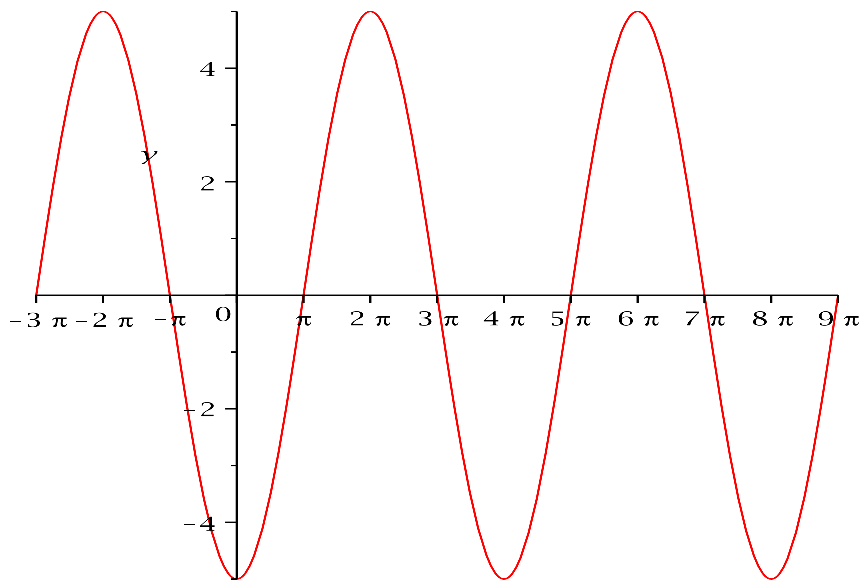
   prove that $\sin(t + \frac{\pi}{2}) = \cos t$.

   Answer:
   Let $A = t$ and $B = \frac{\pi}{2}$ and use the formula above. Then

$$\begin{aligned} \sin(t + \frac{\pi}{2}) &= \sin t \cdot \cos\frac{\pi}{2} + \cos t \cdot \sin\frac{\pi}{2} \\ &= \sin t \cdot 0 + \cos t \cdot 1 \\ &= \cos t\,. \end{aligned}$$

(b) Graph the function $y = 5\sin(\frac{1}{2}(t-\pi))$. Draw two full periods and label tick marks on the horizontal and vertical axes carefully.

Answer: The amplitude of the graph is 5 while the period is $\frac{2\pi}{\frac{1}{2}} = 4\pi$. The phase shift is π so the graph is the usual sine curve shifted to the right by π units.



(c) You and your friend are producing tones by running your fingers around the tops of your partially filled water glasses. You are trying to match pitch with each other but your two tones combined create a strange oscillating tone which pulsates four times every second. If your tone is vibrating at a frequency of 900 Hz, what are the possible frequencies of your friend's tone?

Answer: The strange oscillating tone we hear is beats, so there are four beats per second. This means your friend's tone is 4 Hz away from yours. The possibilites are therefore $900 + 4 = 904$ Hz or $900 - 4 = 896$ Hz.

3. **Fill in the blanks:** Work is **required** to receive partial credit. (4 pts. each)

(a) Increasing the amplitude of a sound by 20 decibels increases the volume of sound by a factor of <u>100</u>.

Answer: Solving the equation $10\log_{10} b = 20$ for b yields $\log_{10} b = 2$ or $b = 10^2 = 100$.

(b) The angle (in radians) that corresponds to going around the unit circle 4.5 times is $\underline{\ 9\pi\ }$. The cosine of this angle is $\underline{\ -1}$.

Answer: Going once around the unit circle (360°) corresponds to an angle of 2π radians. Thus, going around 4.5 times corresponds to an angle of $4.5 \cdot 2\pi = 9\pi$ radians. The cosine of this angle is -1 because the x-coordinate at this point is -1.

(c) A tuning fork produces a sound wave of the form $y = \sin(1000\pi t)$. The frequency of the note you hear is <u>500 Hz</u>.

Answer: Since frequency equals $b/(2\pi)$, we have $(1000\pi)/(2\pi) = 500$ Hz.

(d) A string of length 15 cm and a string of length 20 cm are plucked simultaneously. The musical interval created is a perfect fourth.

Answer: The ratio 15/20 simplifies to 3/4 which is the ratio in the Pythagorean scale for a perfect fourth.

(e) In the overtone series, the **musical interval** between the 5th and 6th notes, that is, between $5f$ and $6f$, is a minor third.

Answer: This follows because the ratio of $6f$ to $5f$ is 6/5 which is a minor third in Just Intonation. Another way to see this is to write out the first 6 notes in the overtone series (starting on any note) and determine the musical interval between the 5th and 6th notes on the staff.

(f) In equal temperament, the number of cents in a major sixth is 900.

Answer: Since there are nine half steps in a major sixth and each half step is equivalent to 100 cents, we obtain 900 cents in a major sixth.

4. **Tuning Systems**

(a) Given that the A above middle C has a frequency of 440 Hz, find the frequency for C♯ above middle C using Just Intonation. (5 pts.)

Answer: 275 Hz. We are looking for the C♯ a half step above middle C. The simplest way to compute the frequency is to drop the octave to A 220 Hz (divide by 2). This gives the A just below middle C. Then, going up a major third results in C♯. In Just Intonation, the ratio for a major third is 5/4 so the answer is $220 \cdot 5/4 = 275$ Hz.

An alternative answer which received full credit is to first go down by a major sixth to middle C. Using the ratio of 5/3, but dividing rather than multiplying, gives $440 \cdot 3/5 = 264$ Hz. Then, since the half step in Just Intonation is 16/15, we raise the pitch by one half step to C♯, obtaining $264 \cdot 16/15 = 281.6$ Hz. This is acceptable because it only involves one half step and avoids any confusion with the two different whole steps of Just Intonation.

(b) Give the exact value of the syntonic comma and give a mathematical derivation demonstrating where it comes from. (6 pts.)

Answer: The syntonic comma is 81/80. It is derived from the ratio of the two different whole steps in Just Intonation, $W = 9/8$ and $W = 10/9$. We have

$$\frac{\frac{9}{8}}{\frac{10}{9}} = \frac{9}{8} \cdot \frac{9}{10} = \frac{81}{80}.$$

(c) Give two advantages of using Just Intonation as a tuning system. (6 pts.)

Answer: Just Intonation is useful for playing in one key (assuming you have tuned the tonic to that key) and particularly nice when playing the major chord of the tonic (1, 3 and 5 of the scale) as well as the IV (subdominant) and V (dominant) major chords. Because the tuning used in Just Intonation matches the overtone series, these chords, which are each in the ratio $4:5:6$, will sound particularly harmonious. Just Intonation also uses ratios with small integers (again coming from the overtone series) and these are easier to remember than other tuning systems.

(d) Give two advantages of using Equal Temperament as a tuning system. (6 pts.)

Answer: Equal Temperament (ET) solves the issue of the spiral of fifths (closing it up to a genuine circle) since 7 octaves in ET is equivalent to 12 perfect fifths (B$\sharp$ = C). Mathematically, this is $(2^{7/12})^{12} = 2^7$. Two half steps equals one whole step in ET and since the octave is equally divided into 12 pieces, it is now possible to switch keys easily using ET.

5. **Irrational and Rational Numbers**

(a) Using Equal Temperament, what number do you multiply the fundamental frequency of a note by to raise it up a perfect fifth? Give the **exact** value (not a decimal). (4 pts.)

Answer: $2^{7/12}$ since a perfect fifth is 7 half-steps and one half-step in Equal Temperament is obtained by multiplying by $2^{1/12}$.

(b) Prove that your answer to part **(a)** is an irrational number. (8 pts.)

Answer: Suppose by contradiction that $2^{7/12}$ is a rational number. Then we have $2^{7/12} = p/q$ for some positive integers p and q. Raising both sides to the 12th power gives $2^7 = p^{12}/q^{12}$. Cross-multiplying gives

$$2^7 \cdot q^{12} \;=\; p^{12}$$

which contradicts the Fundamental Theorem of Arithmetic because the left-hand side will have a prime factorization with an odd number of 2's (7 plus even = odd) while the right-hand side will have an even or zero number of 2's. Therefore, our original assumption of rationality has produced a contradiction and $2^{7/12}$ must be an irrational number.

(c) True or False: The product of two irrational numbers is always irrational. If true, give a proof. If false, provide a counterexample. (5 pts.)

Answer: False. Consider the equation $\sqrt{2} \cdot \sqrt{2} = 2$ which is an irrational number times an irrational number resulting in a rational number $(2 = 2/1)$. Other counterexamples include $\sqrt{2} \cdot (1/\sqrt{2}) = 1$ or $\sqrt{2} \cdot \sqrt{8} = 4$.