MATH 392: Seminar in Celestial Mechanics

Homework Assignment #2

Solution to Problem #3

**Theorem:** Suppose that \( \mathbf{q}_0 \times \mathbf{v}_0 = \mathbf{c} \neq 0 \) where \( \mathbf{q}_0 = \mathbf{q}(0) \) and \( \mathbf{v}_0 = \mathbf{v}(0) \) are the initial position and velocity, respectively. Letting \( c = ||\mathbf{c}|| \), there exists an orthogonal matrix \( A \) and a change of variables \( \mathbf{x} = A\mathbf{q} \) such that the central force problem is converted to

\[
\ddot{\mathbf{x}} = -\frac{f(r)}{r} \mathbf{x}
\]

where \( r = ||\mathbf{x}|| \) and the new angular momentum is simply \((0,0,c)\).

**Proof:**

We will construct the \(3 \times 3\) matrix \( A \) using the vectors \( \mathbf{q}_0 \) and \( \mathbf{c} \). First, we explain the importance of \( A \) being orthogonal. Recall that an orthogonal matrix \( A \) is one for which \( A^T A = A A^T = I \). This is equivalent to having the rows and columns each forming an orthonormal basis for \( \mathbb{R}^3 \) (length one and mutually orthogonal). The key fact about orthogonal matrices is that they preserve lengths of vectors and angles between vectors. In other words, thinking of \( A \) as representing a linear map, the image of vectors under this map preserves lengths and angles.

To see this, write the dot product \( \mathbf{v} \cdot \mathbf{w} \) between any two vectors \( \mathbf{v} \) and \( \mathbf{w} \) as

\[
\mathbf{v} \cdot \mathbf{w} = \mathbf{v}^T \mathbf{w}
\]

interpreting the resulting \(1 \times 1\) matrix as a scalar. Then we have, for any orthogonal matrix \( A \),

\[
||A\mathbf{v}||^2 = (\mathbf{v} \cdot \mathbf{v}) = (A\mathbf{v})^T (A\mathbf{v}) = \mathbf{v}^T A^T A \mathbf{v} = \mathbf{v}^T \mathbf{v} = ||\mathbf{v}||^2
\]

so that \( ||A\mathbf{v}|| = ||\mathbf{v}|| \) and the length of \( \mathbf{v} \) is unchanged under the linear map \( A \). Furthermore, using the fact that \( \mathbf{v} \cdot \mathbf{w} = ||\mathbf{v}|| ||\mathbf{w}|| \cos(\theta) \) where \( \theta \in [0, \pi] \) is the angle between \( \mathbf{v} \) and \( \mathbf{w} \), we see that

\[
\cos(\theta) = \frac{(A\mathbf{v}) \cdot (A\mathbf{w})}{||A\mathbf{v}|| ||A\mathbf{w}||} = \frac{\mathbf{v} \cdot \mathbf{w}}{||\mathbf{v}|| ||\mathbf{w}||}.
\]

This shows that the angle between \( A\mathbf{v} \) and \( A\mathbf{w} \) is the same as the angle between \( \mathbf{v} \) and \( \mathbf{w} \), although the orientation may be reversed if \( \det(A) = -1 \).

Using these properties, it is easy to see why the change of variables \( \mathbf{x} = A\mathbf{q} \) leads to the identical Kepler problem but with \( \mathbf{x} \) replacing \( \mathbf{q} \). Since \( A^T = A^{-1} \), we have

\[
\ddot{\mathbf{x}} = A \dot{\mathbf{q}} = A \left( -\frac{f(r)}{r} \right) \mathbf{q} = -\frac{f(r)}{r} A A^T \mathbf{x} = -\frac{f(r)}{r} \mathbf{x}
\]

where \( r = ||\mathbf{q}|| = ||A^T \mathbf{x}|| = ||\mathbf{x}|| \) is unchanged since the orthogonal matrix \( A^T \) preserves lengths.

Since \( \dot{\mathbf{x}} = A \dot{\mathbf{q}} \), the angular momentum in the new coordinates will be the vector

\[
\mathbf{x}(0) \times \dot{\mathbf{x}}(0) = (A\mathbf{q}(0)) \times (A \dot{\mathbf{q}}(0)) = (A\mathbf{q}_0) \times (A \mathbf{v}_0).
\]

Since \( A \) is orthogonal, the new angular momentum will have the same length \( c \). This follows from

\[
||\mathbf{x}(0) \times \dot{\mathbf{x}}(0)|| = ||(A\mathbf{q}_0) \times (A \mathbf{v}_0)|| = ||A\mathbf{q}_0|| ||A \mathbf{v}_0|| \sin \theta = ||\mathbf{q}_0|| ||\mathbf{v}_0|| \sin \theta = ||\mathbf{q}_0 \times \mathbf{v}_0|| = c
\]
where the angle θ between \( \mathbf{q}_0 \) and \( \mathbf{v}_0 \) is unchanged under the map \( A \). Thus if we can construct \( A \) so that the first two coordinates of \( \mathbf{x}(0) \times \hat{\mathbf{x}}(0) \) are both zero, the third coordinate will either be \( c \) or \(-c\) so that the length remains fixed at \( c \).

To have the cross product of two vectors lying in the \( z \)-direction only, we want the vectors to be in the \( xy \)-plane. Thinking of how matrix multiplication works, it follows that we want the third row of \( A \) to be orthogonal to the vectors \( \mathbf{q}_0 \) and \( \mathbf{v}_0 \). But \( c \) is such a vector! Choose the third row of \( A \) to be the unit vector \( \mathbf{c}/c \). The remaining two rows of \( A \) must be orthogonal to \( c \) and to themselves. A natural choice is to choose \( \mathbf{q}_0 \) (orthogonal to \( c \)) and \( c \times \mathbf{q}_0 \) (orthogonal to both \( c \) and \( \mathbf{q}_0 \) by the definition of the cross product). The reason for choosing \( c \times \mathbf{q}_0 \) and not \( \mathbf{q}_0 \times c \) will be made clear in a moment. Note that \( ||c \times \mathbf{q}_0|| \) is the same as \( c ||\mathbf{q}_0|| \) since the vectors are orthogonal.

In sum, the three rows of \( A \) are given by

\[
A = \begin{bmatrix}
\frac{\mathbf{q}_0}{||\mathbf{q}_0||} \\
\frac{c \times \mathbf{q}_0}{||\mathbf{q}_0||} \\
c/c
\end{bmatrix}
\]

(each vector is really transposed so that it becomes a row rather than a column vector). By construction, \( A \) is an orthogonal matrix with \( A\mathbf{q}_0 \) and \( A\mathbf{v}_0 \) lying in the \( xy \)-plane. This implies that

\[
A\mathbf{q}_0 \times A\mathbf{v}_0 = [0 \ 0 \ \pm c]^T.
\]

It remains to show that the third component of this vector is in fact just \( c \). This can be done by showing that \( \det(A) = 1 \) or more directly by actually calculating the cross product.

Interpreting matrix multiplication via the dot product, we see that \( A\mathbf{q}_0 = [r_0 \ 0 \ 0]^T \) where \( r_0 = ||\mathbf{q}_0|| \). The vector \( A\mathbf{v}_0 \) is

\[
A\mathbf{v}_0 = \begin{bmatrix}
\frac{\mathbf{q}_0 \times \mathbf{v}_0}{r_0} \\
\frac{c}{r_0} \\
0
\end{bmatrix}
\]

where the second component follows using the vector identity from question #2:

\[
\left(\frac{\mathbf{c}}{c} \frac{\mathbf{q}_0}{r_0}\right) \cdot \mathbf{v}_0 = \frac{c}{c} \left(\frac{\mathbf{q}_0}{r_0} \times \mathbf{v}_0\right) = \frac{c}{r_0} = \frac{c}{r_0}.
\]

Thus, computing the cross product of these two vectors gives

\[
A\mathbf{q}_0 \times A\mathbf{v}_0 = \begin{bmatrix}
\frac{\mathbf{q}_0 \times \mathbf{v}_0}{r_0} \\
r_0 \quad 0 \\
r_0 \quad 0
\end{bmatrix}
\begin{bmatrix}
0 \\
0 \\
c
\end{bmatrix}
\]

as desired. \( \square \)