Seminar in Mathematics and Climate

Homework Assignment #4

Due Thurs., March 15, start of class

You should write up solutions neatly to all problems, making sure to show all your work. A nonempty subset will be graded. You are encouraged to work on these problems with other classmates, and it is ok to use internet sources for help if it's absolutely necessary (with proper citation); however, the solutions you turn in should be your own work and written in your own words.

Note: Please list the names of any students or faculty who you worked with on the assignment.

1. In this exercise you will search for bifurcations in the Budyko-Widiasih Energy Balance Model with a two-step albedo function. Recall that in addition to Budyko's equation, a second differential equation is used to model the movement of the ice line. The key idea is to use Budyko's equation to compute the equilibrium temperature at the ice line η and then change η depending on whether this temperature is above or below $-10°$C. Define $\alpha_0 = (\alpha_w + \alpha_s)/2$ to be the average of the albedo rates for land/water and snow/ice.

   The ODE governing the movement of the ice line is

$$\begin{aligned} \frac{d\eta}{dt} &= \epsilon h(\eta) \\ h(\eta) &= \frac{Q}{B+C}\left[s(\eta)(1-\alpha_0) + \frac{C}{B}\left(1-\alpha_s + (\alpha_s - \alpha_w)(1.241\eta - 0.241\eta^3)\right)\right] - \frac{A}{B} + 10, \end{aligned}$$

   where $\epsilon > 0$ is a small parameter measuring the rate of movement of the ice line. A typical graph of $h(\eta)$ is shown below, indicating two equilibria ice lines at $\eta_1 \approx 0.2561527$ (unstable) and $\eta_2 \approx 0.9394721$ (stable). We will always assume that $0 \leq \eta \leq 1$.

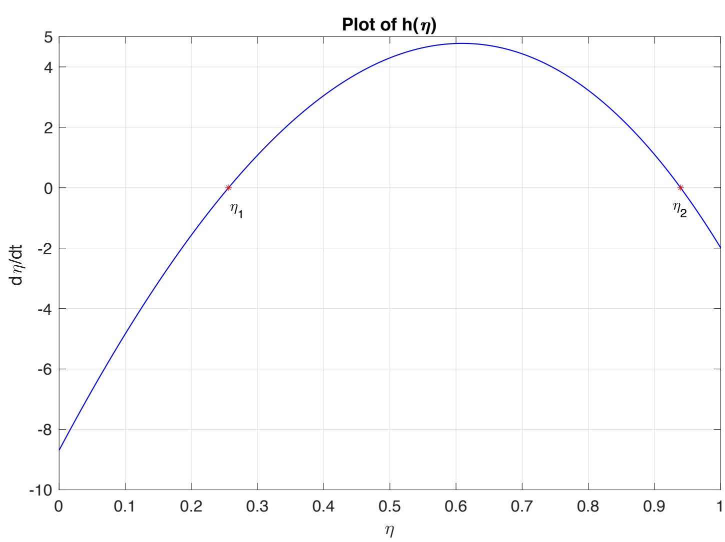

Figure 1: A graph of $h(\eta)$ indicating a small stable ice cap at $\eta = \eta_2$ and a large unstable ice cap at $\eta = \eta_1$. Parameter values: $Q = 342, A = 202, B = 1.9, C = 3.04, \alpha_w = 0.32, \alpha_s = 0.62$.

(a) The parameter A can be treated as a proxy for the amount of CO_2 in the atmosphere. In terms of the Budyko model, explain why A and the amount of CO_2 in the atmosphere are *inversely* related, that is, if A increases, then the amount of CO_2 decreases, and vice versa.

(b) What effect does varying A have on the graph of $h(\eta)$? Be specific.

(c) Fix the parameters $Q = 342, B = 1.9, C = 3.04, \alpha_w = 0.32, \alpha_s = 0.62$, but allow A to vary. Use technology (e.g., Matlab or Maple) to find the value of A (six decimal places) where a saddle-node (tangent) bifurcation occurs. Sketch the graph of h at the bifurcation, and sketch phase lines before, at, and after the bifurcation.

(d) Fixing the parameters as in part **(c)**, find the value of A (six decimal places) where the small stable ice cap suddenly disappears, and there is only one large ice cap remaining. *Hint:* This is a result of assuming that $\eta \leq 1$.

(e) Fixing the parameters as in part **(c)**, find the value of A (six decimal places) where the large unstable ice cap suddenly disappears, and then no equilibria remain. *Hint:* This is a result of assuming that $\eta \geq 0$.

(f) Using your results from all of the above, sketch a bifurcation diagram for the Budyko-Widiasih EBM as A varies. Put A on the horizontal axis and η on the vertical axis. Stable equilibria should be indicated with a solid curve, while unstable equilibria should be dashed. Indicate all of the key A-values in your diagram and be sure to include plenty of phase lines. Write a one paragraph summary explaining your figure in terms of the change in the overall climate as A varies, paying particular attention to behavior before and after bifurcations (use words like tipping point, Snowball Earth, hothouse, stable ice cap, etc.).

2. Recall the "dimensionless" version of Stommel's one-box model derived in class,

$$\begin{aligned} x' &= \delta(1 - x) \\ y' &= 1 - y\,. \end{aligned}$$

Here, x and y are the salinity and temperature ratios, respectively, in relation to the surrounding medium ($x = S/S^*, y = T/T^*$) and $\delta = d/c$ is a positive parameter comparing the rates between salinity and temperature. The new "time" variable is τ, that is, $x' = dx/d\tau$.

(a) Find the general solution to this system in terms of the initial conditions $x_0 = x(0)$ and $y_0 = y(0)$. In other words, find $x(\tau)$ and $y(\tau)$. Your answer will contain x_0 and y_0 in it.

(b) What is the equilibrium point for this system? Explain why it is called a *sink*. What does this point represent physically in the model?

(c) Find the particular solution to the system satisfying $x_0 = 0, y_0 = 0$.

(d) Sketch the phase plane diagrams for the system in the xy-plane for $\delta = 1/6, \delta = 1$, and $\delta = 3$.

(e) Using a different color, sketch the particular solution from part **(c)** on each of your phase plane diagrams in part **(d)**.