MATH 392-02: Seminar in Complex Analytic Dynamics

Homework Assignment #6 (last one!)

DUE DATE: Fri., April 20, start of class.

Homework should be turned in at the BEGINNING OF CLASS. You should write up solutions neatly to all problems, making sure to show all your work. For this assignment you may work in a group of up to three students. Please cite any references (websites, texts, etc.) that you may have used for assistance with the assignment.

1. Prove the "Sunshine" Theorem: For any complex polynomial p,

$$J(p) \;=\; \partial K(p) \;=\; \partial B(\infty).$$

   In other words, show that the Julia set is equal to the boundary of the filled Julia set which in turn, is equal to the basin of attraction of infinity.

2. In this problem you will prove that any polynomial of degree $d \geq 2$ is conjugate to the map $z \mapsto z^d$ on some neighborhood of ∞.

   a) Suppose that $g(z) = z^k R(z)$ where $k \in \mathbb{N}$, $k \geq 2$ and $R(z)$ is a rational function such that $R(0) \neq \infty$ and $R(0) \neq 0$. Show that $g'(z)$ is of the same form as $g(z)$, but with $k-1$ replacing k.

   b) Suppose that $g(z) = zR(z)$ where $R(z)$ is a rational function such that $R(0) \neq \infty$ and $R(0) \neq 0$. Show that $g'(0) \neq 0$.

   c) Let $p(z)$ be a polynomial of degree $d \geq 2$. Using the conjugate map $g(z) = 1/p(1/z)$, show that

$$p'(\infty) \;=\; p''(\infty) \;=\; p'''(\infty) \;=\; \cdots \;=\; p^{(d-1)}(\infty) \;= 0$$

   but $p^{(d)}(\infty) \neq 0$.

   d) Using a theorem from class, show that any polynomial of degree $d \geq 2$ is conjugate to the map $z \mapsto z^d$ on some neighborhood of ∞.

3. In this problem you will prove that $Q_0(z) = z^2$ is analytically conjugate to $Q_{-2}(z) = z^2 - 2$ via the conjugacy $h(z) = z + 1/z$.

   a) Consider $h(z) = z + 1/z$ defined on the domain $V = \{z : |z| > 1\}$. Show that h is a homeomorphism on V and that $h(V) = \mathbb{C} - [-2, 2]$.

   b) Check that $h \circ Q_0 = Q_{-2} \circ h$ and conclude that Q_0 on V is analytically conjugate to Q_{-2} on $\mathbb{C} - [-2, 2]$.

   c) Find the sets $B(\infty)$, $J(Q_{-2})$ and $K(Q_{-2})$ for the function $Q_{-2}(z)$.

   d) **Bonus Question:** Show that h maps rays in V perpendicular to the unit circle to hyperbolas (including possibly degenerate hyperbolas) with vertices in $[-2, 2]$. Also show that h maps circles centered at the origin in V to ellipses centered at the origin in $\mathbb{C} - [-2, 2]$.

4. Consider $Q_c(z) = z^2 + c$ where $|c| > 2$.

a) Show that if $|z| \geq |c| > 2$, then there exists a $\delta > 0$ such that

$$|Q_c^n(z)| \ > \ (1+\delta)^n |z| \quad \forall n \in \mathbb{N}.$$

b) Show that if $|z| \geq |c| > 2$, then $z \in B(\infty)$ for Q_c.

c) Show that the Mandelbrot set is contained inside the solid disk $\{c : |c| \leq 2\}$. Explain why there can be no *smaller* disk (centered at the origin) containing the Mandelbrot set.

5. Show that the period two-bulb of the Mandelbrot set is the interior of the circle of radius 1/4 centered at $c = -1$. In other words, show that the set of parameter values c for which Q_c has an attracting period 2-cycle is equal to the interior of the circle of radius 1/4 centered at $c = -1$.

6. Prove that the Mandelbrot set is symmetric with respect to the real axis by first showing that $Q_c(z) = z^2 + c$ and $Q_{\overline{c}}(z) = z^2 + \overline{c}$ are conjugate. Note that your conjugacy h will *not* be analytic but it will still be a homeomorphism (check this). Then explain why this implies that the Mandelbrot set is symmetric with respect to the real axis.