MATH 392-02: Seminar in Complex Analytic Dynamics

Homework Assignment #4

DUE DATE: Wed., March 21, start of class.

Homework should be turned in at the BEGINNING OF CLASS. You should write up solutions neatly to all problems, making sure to show all your work. You are encouraged to work on these problems with other classmates, although the solutions you turn in should be your **own** work. Please cite any references (websites, texts, etc.) that you may have used for assistance with the assignment.

Note: Please list the names of any students or faculty who you worked with on the top of the assignment.

1. Consider the linear map $L_\lambda(z) = \lambda z$ where $|\lambda| = 1$. Show that ∞ is in the Fatou set of L_λ by rigorously showing that the conjugate map is equicontinuous on any neighborhood of the origin. This completes the proof that $F(L_\lambda) = \overline{\mathbb{C}}$ when $|\lambda| = 1$.

2. We say that a function $f : X \mapsto Y$ is ***uniformly continuous*** if $\forall \epsilon > 0$, there exists a $\delta > 0$ such that if $x_1, x_2 \in X$ and $|x_1 - x_2| < \delta$, then $|f(x_1) - f(x_2)| < \epsilon$. The key idea here is that δ must be chosen *independent* of x_1 and x_2, and must work for all sufficiently close x-values. A function that is uniformly continuous is certainly continuous, but the converse is false. This definition is valid for either real or complex functions.

   a) Show that the real function $f(x) = x^2$ is uniformly continuous on any closed interval $[a, b] \subset \mathbb{R}$.

   b) Explain why the real function $f(x) = x^2$ is not uniformly continuous on all of $\mathbb{R}$.

3. For this problem you will need the fact that a continuous function defined on (or restricted to) a compact set is uniformly continuous.

   Suppose that $f(z)$ and $g(z)$ are analytically conjugate with conjugacy $h(z)$. Using the Arzela-Ascoli Theorem, prove that if z_0 is in the Fatou set for f, then $h(z_0)$ is in the Fatou set of g. In other words, show that h maps the Fatou set for f to the Fatou set for g.

   Note: It is not sufficient to just claim that attractors go to attractors under conjugacy, since the Fatou set may contain neutral fixed and periodic points, as well as points in a neighborhood of neutral fixed and periodic points. By the Neutral Fixed Point Theorem, on such a neighborhood the map is conjugate to a rotation and is not "attracting."

4. (Inspired by Liz Bolduc) Suppose that z_0 is a neutral fixed point in the Fatou set of $R(z)$ and let U be the largest simply connected open set in $F(R)$ containing z_0. Use the Neutral Fixed Point Theorem and the Riemann Mapping Theorem to prove that $R(U) = U$.

   Recall that the set $R(U) = \{w \in \mathbb{C} : \exists z \in U \text{ with } R(z) = w\}$.

5. Suppose that z_0 is a neutral fixed point for $R(z)$ and that there exists a sequence of periodic points $\{z_n\}$ converging to z_0. Is z_0 in the Fatou set or Julia set of R? Explain.

6. Each of the following functions has a neutral fixed point. Find the neutral fixed point and then determine whether it is in the Fatou set or Julia set of the given function.

   a) $Q(z) = z^2 - \dfrac{3}{4}$

   b) $R(z) = z^2 + \dfrac{1}{4} + \dfrac{i}{2}$

   c) $p(z) = 4i + [\cos(2\sqrt[4]{7}\,\pi) + i\sin(2\sqrt[4]{7}\,\pi)](z - 4i) + 2012(z - 4i)^5$