MATH 392-02: Seminar in Complex Analytic Dynamics

Homework Assignment #4

Solution to Problem #3

Suppose that $f(z)$ and $g(z)$ are analytically conjugate with conjugacy $h(z)$. Using the Arzela-Ascoli Theorem, prove that if z_0 is in the Fatou set for f, then $h(z_0)$ is in the Fatou set of g. In other words, show that h maps the Fatou set for f to the Fatou set for g.

Proof: Let z_0 be an arbitrary element in the Fatou set of $f(z)$. By definition, there exists a neighborhood U containing z_0 such that $\{f^n\}$ is a normal family on U. Let $U' = h(U)$. Since $h(z)$ is an analytic function, U' is an open neighborhood about $h(z_0)$. Let V' be an arbitrary compact subset of U'. We will show that $\{g^n\}$ is an equicontinuous family on V'. By the Arzela-Ascoli Theorem, this will prove that $\{g^n\}$ is a normal family on U' and consequently, $h(z_0)$ belongs to the Fatou set of $g(z)$.

Claim: $\{g^n\}$ is an equicontinuous family on V'.

Let $\epsilon > 0$ be given. We must find a $\delta > 0$ such that if $w_1, w_2 \in V'$ and $|w_1 - w_2| < \delta$, then $|g^n(w_1) - g^n(w_2)| < \epsilon \quad \forall\, n \in \mathbb{N}$.

First, consider the set $h^{-1}(V') = V$. Since h^{-1} is continuous, V is a compact set inside U (see Figure 1). Since $\{f^n\}$ is a normal family on U, we know by the Arzela-Ascoli Theorem that $\{f^n\}$ is an equicontinuous family on V. This means that points close in V will remain close for all iterates of f. The tricky part is figuring out where all of these iterates lie so that we may extend the continuity of h to uniform continuity on some compact set. We don't know the domain of the conjugacy h; in fact, the domain of a conjugacy is often restricted to just the Fatou set. Often it fails to be one-to-one on the Julia set, for example. However, we only require that h be continuous on a compact set in order to assume that it is uniformly continuous.

Consider the set

$$\mathcal{V} \;=\; \bigcup_{n=1}^{\infty} f^n(V)$$

which, since f^n is continuous for each n, is an infinite union of compact sets. Since the Fatou set is invariant, we know that $\mathcal{V}$ is a subset of the Fatou set of f. However, the infinite union of compact sets may *not* be compact! For example, consider the set

$$\bigcup_{n=1}^{\infty} \{z : |z| \le 1 - \frac{1}{n}\},$$

which is the infinite union of compact sets. But this set equals the open unit disk $\{z : |z| < 1\}$ which is an *open* set. Thus, we must consider the closure of $\mathcal{V}$.

Define $W = \overline{\mathcal{V}}$. W is clearly a closed set and it is bounded on the Riemann sphere. Thus, W is a compact set and we can extend the conjugacy h (if need be) so that h is analytic on the interior of W and continuous on the boundary of W. Note that the boundary likely contains points in the Julia set of f.

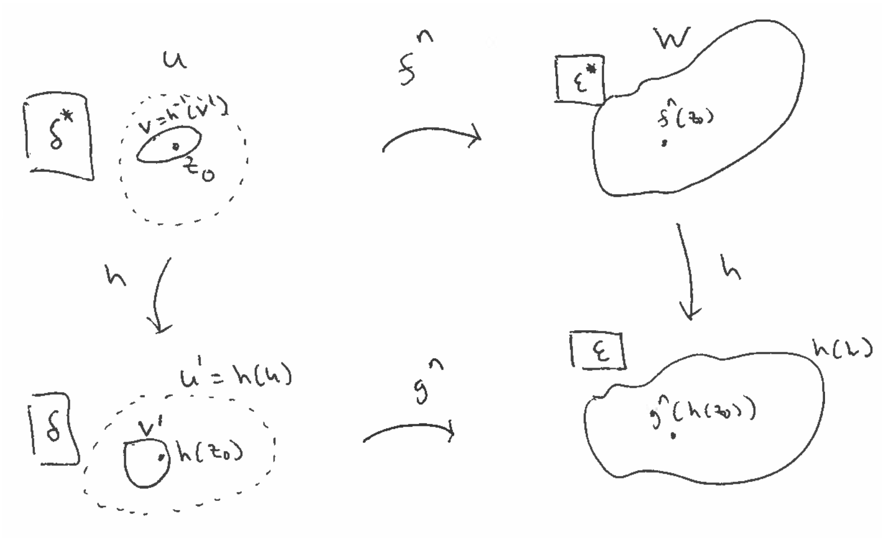

Figure 1: The commutative diagram indicating the conjugacy and the spaces where $\epsilon, \epsilon^*, \delta, \delta^*$ reside.

Now we can prove our claim through a series of implications involving the definitions of equicontinuity and uniform continuity. We begin with W (see Figure 1). Since h is continuous on W and W is compact, there exists an $\epsilon^* > 0$ such that

$$|z_1 - z_2| < \epsilon^* \text{ and } z_1, z_2 \in W \implies |h(z_1) - h(z_2)| < \epsilon. \tag{1}$$

Since $\{f^n\}$ is an equicontinuous family on V, there exists a $\delta^* > 0$ such that

$$|v_1 - v_2| < \delta^* \text{ and } v_1, v_2 \in V \implies |f^n(v_1) - f^n(v_2)| < \epsilon^* \; \forall n \in \mathbb{N}. \tag{2}$$

Finally, since V' is compact and h^{-1} is continuous, h^{-1} is uniformly continuous on V'. Therefore, there exists a $\delta > 0$ such that

$$|w_1 - w_2| < \delta \text{ and } w_1, w_2 \in V' \implies |h^{-1}(w_1) - h^{-1}(w_2)| < \delta^*. \tag{3}$$

This is the δ we seek. We now check that our δ works. Suppose that w_1 and w_2 are points in V' which are less than δ apart. Then, we have

$$\begin{aligned}
|w_1 - w_2| < \delta &\implies |h^{-1}(w_1) - h^{-1}(w_2)| < \delta^* \quad \text{by (3)} \\
&\implies |f^n(h^{-1}(w_1)) - f^n(h^{-1}(w_2))| < \epsilon^* \quad \forall n \in \mathbb{N} \quad \text{by (2)} \\
&\implies |h(f^n(h^{-1}(w_1))) - h(f^n(h^{-1}(w_2)))| < \epsilon \quad \forall n \in \mathbb{N} \quad \text{by (1)} \\
&\implies |g^n(w_1) - g^n(w_2)| < \epsilon \quad \forall n \in \mathbb{N} \quad \text{since } g^n = h \circ f^n \circ h^{-1}.
\end{aligned}$$

This shows that $\{g^n\}$ is an equicontinuous family on V' and thus, $\{g^n\}$ is a normal family on U' so that $h(z_0)$ is in the Fatou set of $g(z)$. $\square$.