MATH 392-02: Seminar in Complex Analytic Dynamics

Homework Assignment #3

DUE DATE: Fri., Feb. 24, start of class.

Homework should be turned in at the BEGINNING OF CLASS. You should write up solutions neatly to all problems, making sure to show all your work. You are encouraged to work on these problems with other classmates, although the solutions you turn in should be your **own** work. Please cite any references (websites, texts, etc.) that you may have used for assistance with the assignment.

Note: Please list the names of any students or faculty who you worked with on the top of the assignment.

1. Consider the *doubling map*, $d(x) = 2x \bmod 1$ on $[0, 1)$, which is equivalent to the piecewise function
$$d(x) = \begin{cases} 2x & \text{if } 0 \le x < 1/2 \\ 2x - 1 & \text{if } 1/2 \le x < 1. \end{cases}$$
The dynamics of this map are equivalent to the doubling map $f(\theta) = 2\theta$ on the unit circle.

   a) Draw a graph of the second iterate $d^2(x)$.

   b) Carefully describe and draw a graph of the nth iterate $d^n(x)$.

   c) Using your answer to part **b)**, prove that $d(x)$ is a chaotic dynamical system by verifying that all three properties are satisfied.

2. Suppose that $f : A \mapsto A$ and $g : B \mapsto B$ are continuous functions that are conjugate to each other with conjugacy h. Show that if f is topologically transitive, then g is also topologically transitive. This result finishes the proof of the theorem stating that if the continuous functions f and g are conjugate, and f is chaotic, then g is also chaotic.

   Hint: You may use the fact that if U is an open set in B, then $h^{-1}(U)$ is an open set in A. This fact holds for any continuous function h.

3. Consider the sequence of complex functions $f_n(z) = nz^n$ on the domain $\overline{\mathbb{C}}$.

   a) Find the pointwise limit
$$\lim_{n\to\infty} f_n(z) = f(z),$$
that is, give an explicit expression for the limit function $f(z)$ on $\overline{\mathbb{C}}$.

   b) Let $\overline{\mathbb{D}} = \{z : |z| \le 1\}$ be the solid unit disk. Is the convergence of $\{f_n(z)\}$ to $f(z)$ on the set $\overline{\mathbb{D}}$ uniform? Explain.

   c) Let $\mathbb{D} = \{z : |z| < 1\}$ be the interior of the unit disk. Is the convergence of $\{f_n(z)\}$ to $f(z)$ on the set $\mathbb{D}$ uniform? Explain.

4. The Fatou and Julia sets for z^d:

a) Find the Fatou and Julia sets of $f(z) = z^3$ and give a rigorous proof that justifies your answer.

b) Find the Fatou and Julia sets of $f(z) = z^d$, where $d \in \mathbb{N}$ and $d \geq 4$ (no proof required).

c) What are the dynamics on the Julia set for $f(z) = z^d$? In other words, when restricting $f(z)$ to the Julia set, what function is being iterated and how does it depend on the degree d?

5. Prove that if z_0 is a repelling fixed point for $R(z)$, then z_0 is in the Julia set $J(R)$ of $R(z)$.

Hint: Use the Repelling Fixed Point Theorem.