Seminar in Complex Analytic Dynamics

MATH 392-02

Sample Midterm Exam Questions

1. Give precise definitions of the following concepts:

   (a) A super-attracting period n-cycle

   (b) A homeomorphism

   (c) An equicontinuous family of functions $\{f_i : U \mapsto \overline{\mathbb{C}}\}$

2. State and prove the Attracting Fixed Point Theorem.

3. Give an example of a complex dynamical system that has a dense orbit. Explain.

4. Find all fixed points and period 2-cycles of the function $g(z) = -1/z$. Classify them as attracting, repelling, neutral or super-attracting.

5. Consider the family of rational functions

$$f_{a,b}(z) = \frac{az^2}{bz+1}$$

   where a, b are two complex parameters.

   (a) Find all the fixed points in $\mathbb{C}$ of $f_{a,b}$ (your answer will depend on a and b).

   (b) Find conditions on a and b that guarantee all the fixed points from part **(a)** are attracting.

   (c) Show that ∞ is a fixed point.

   (d) Find conditions on a and b that guarantee that ∞ is a neutral fixed point.

6. Give an example of a linear function where the Fatou set is equal to the following set:

   (a) $\overline{\mathbb{C}} - \{0\}$

   (b) $\overline{\mathbb{C}}$ (the entire extended complex plane).

7. **(a)** Suppose that $\{f_n(z)\}$, $f_n : U \mapsto \overline{\mathbb{C}}$, is a sequence of functions defined on a common domain U. Give a precise definition of what it means for $\{f_n(z)\}$ to converge uniformly to the constant function $f(z) = \infty$. You should do this without using the conjugacy $h(z) = 1/z$.

   (b) Use your definition to prove that the sequence $\{f_n(z)\}$, where $f_n(z) = n(z^2 - n)$ converges uniformly to the constant function $f(z) = \infty$ on any compact subset of $\overline{\mathbb{C}}$.

8. TRUE or FALSE. If the statement is true, provide a **proof**. If the statement is false, provide a **counterexample** or explain why the statement is false.

   (a) If f and g are analytically conjugate, and f is a chaotic dynamical system, then periodic points are dense for g.

   (b) If a sequence of complex analytic functions converges pointwise on some compact set U, then the limit function is also analytic.

   (c) It is possible for the Julia set of a rational map of degree 2 to be the set $\{z : 1 \leq |z| < 2\}$.