FINAL PROJECT

Seminar in Complex Analytic Dynamics

Spring 2012

The final assignment of the course is to complete a substantial project focusing on some particular aspect or application related to the course material. Your project will consist of both a typed report (roughly 10 pages) and an in-class presentation (25 minutes) during the final week of class. Your report can be written using Maple (which has nice word processing for mathematical symbols) or a regular word processing program with a hand-written appendix for mathematical formulae. You will be allowed to work in small groups (2–3 people) for the project although it is expected that each member will contribute equally. The final project is worth 30% of your total course grade.

Timeline and Due Dates:

- March 30: Brief description of final project topic, including at least three references
- April 20: Brief progress report detailing status of the project, including results and further lines of inquiry. By this date you should have met with me at least once to discuss the content of your report and the plan for your presentation.
- April 27: Title of Final Project along with names of group members
- May 2–7: Project Presentations (25 minutes)
- May 7: Final Report due (typed, roughly 10 pages)

The aim of this project is for you to explore in greater detail a specific topic in complex dynamics. Ideally, you will apply mathematical knowledge gained from this course, as well as others, to make an in-depth investigation of your topic. This may involve reading research papers or textbooks, presenting the results, and doing some actual mathematics, perhaps proving a few theorems along the way. This is not expected to be a ground-breaking research paper leading to publication, but rather a chance for you to delve deeper into a topic employing your well-developed mathematical abilities. Some sample topics are suggested below, along with sample resources. In many cases, the exercises given in Devaney's text in Chapter 18 provide an excellent starting point for your analysis. Feel free to suggest your own topic if there is something different you would like to investigate. In particular, any of the major theorem(s) from complex analysis we have not proven in class could lead to an interesting project.

Caution: Be careful when using material found on the Internet. For example, some of the information on *Wikipedia* is correct and some is not. Be sure to check your findings on the Internet thoroughly by confirming them with at least two independent, published (ie. peer-reviewed) sources.

Sample Topics:

1. **Newton's Method as a Complex Dynamical System**

   There are equations that computers have a lot of trouble solving. For example, when applying Newton's method to certain polynomials, the iterative method may actually fail to find a root on an entire region of initial guesses. Such "bad" polynomials are interesting to study from a dynamical systems perspective. Interesting Julia sets and figures in the parameter plane arise from applying Newton's method to complex polynomials.

Sample Resources: Chapter 13 and Section 18.5 of Devaney's text as well as Plate 39 (nice figure). "Newton's versus Halley's method: A dynamical systems approach," G. E. Roberts and J. Horgan-Kobelski, *International Journal of Bifurcation and Chaos*, Vol. 14, No. 10 (2004), 3459–3475.

2. **The Dynamics of λe^z**

   How do the dynamics of the family of exponential functions $E_\lambda(z) = \lambda e^z$ differ from the quadratic family $Q_c(z) = z^2 + c$? Compare and contrast the dynamical features of E_λ with those studied in class for rational maps of degree $d \geq 2$. Is their an analog of the Mandelbrot set? What type of point is ∞? What are the interesting topological features of the Julia set? How can we study the parameter plane? Write or use a computer program to draw the Fatou and Julia sets for different cases.

   Sample Resources: Section 18.3 and Plates 30–33 (figures) of Devaney's text. "Hairs for the complex exponential family," C. Bodelón, R. L. Devaney, M. Hayes, G. E. Roberts, L. Goldberg and J. Hubbard, *International Journal of Bifurcation and Chaos*, Vol. 9, No. 8 (1999), 1517–1534.

3. **The Dynamics of Trig. Functions**

   Consider the families of trigonometric functions $S_\lambda(z) = \lambda \sin z$, $C_\lambda(z) = \lambda \cos z$ and $T_\lambda(z) = \lambda \tan z$. How do the dynamics of these families differ from the quadratic family $Q_c(z) = z^2 + c$? Compare and contrast the dynamical features of these families with those studied in class for rational maps of degree $d \geq 2$. Is there an analog of the Mandelbrot set? What type of point is ∞? What are the interesting topological features of the Julia set? How can we study the parameter plane? Write or use a computer program to draw the Fatou and Julia set for different cases.

   Sample Resources: Section 18.4 of Devaney's text. Plates 25–29 and 34–35 in Devaney's text show some nice figures. "Parabolic perturbation of the family $\lambda \tan z$," L. Keen and S. Yuan, *Complex dynamics*, Contemp. Math., Vol. 396, Amer. Math. Soc., Providence, RI, (2006), 115–128.

4. **The Tricorn**

   Consider the non-analytic family of functions $A_c(z) = \overline{z}^2 + c$. Due to the presence of $\overline{z}$, these functions are not analytic. However, the second iterate $A_c^2(z)$ is analytic since it is a fourth-degree polynomial. The critical point of A_c turns out to be $z_0 = 0$ and it plays a similar role here as it did in the quadratic family $Q_c(z) = z^2 + c$. The analog of the Mandelbrot set was dubbed by John Milnor as a *tricorn*, because it resembles a tricornered hat. There are now "multicorns" and "unicorns" in complex dynamics. Explore the dynamics and derivation of the tricorn for the family A_c, comparing and contrasting your findings with the quadratic family Q_c. Write or use a computer program to draw and investigate the tricorn.

   Sample Resources: Section 18.1 and Plates 36–38 (figures) of Devaney's text. "Connectedness of the tricorn," S. Nakane, *Ergodic Theory Dynam. Systems*, Vol. 13, No. 2 (1993), 349–356. "On multicorns and unicorns. I. Antiholomorphic dynamics, hyperbolic components and real cubic polynomials," S. Nakane and D. Schleicher, *Internat. J. Bifur. Chaos Appl. Sci. Engrg.*, Vol. 13, No. 10 (2003), 2825–2844.

5. **Iterating Cubic Polynomials**

The study of the dynamics of cubic polynomials turns out to be considerably more difficult than the study of quadratic functions. The primary reason is that there are now two critical points rather than just one. The critical points may have bounded orbits or they may escape to ∞. However, a new case presents itself here; it is possible for one critical point to escape and for the other critical point to remain bounded. This leads to a much more complicated parameter plane. In fact, since the general cubic is conjugate to

$$C_{a,b}(z) \;=\; z^3 + az + b,$$

the true parameter space is really $\mathbb{C}^2$, a four-dimensional space. Nonetheless, much of the analysis from the quadratic family is applicable here, although the results are more complicated and new dynamical features arise. Explore the family $C_{a,b}$ paying attention to interesting bifurcations that arise as the parameters vary. Write or use a computer program to draw the Fatou and Julia set for different cases.

Sample Resources: Section 18.2 of Devaney's text. "Cubic polynomial maps with periodic critical orbit," J. Milnor, *Complex Dynamics*, A. K. Peters, Wellesley, MA, 2009, 333–411. "Remarks on iterated cubic maps," J. Milnor, *Experiment. Math.*, Vol. 1, No. 1, (1992), 5–24.

6. **Sierpinski Carpets and Gaskets as Julia Sets**

There are some very interesting topological features of Julia sets that arise for the family

$$f_\lambda \;=\; z^2 + \frac{\lambda}{z^d}$$

where $\lambda \in \mathbb{C}$ is a complex parameter and $d = 1$ or $d = 2$. This family is a type of *singular perturbation*, where turning on the parameter λ (starting with $\lambda = 0$) modifies the simple case z^2 into a complicated rational map with a pole of order d at the origin. Instead of a fixed point at the origin, we now have a "trap-door" near the origin, where points escape toward the super-attracting fixed point at ∞. If the orbit of the critical points goes to ∞, then the Julia set is either a Cantor set or a Sierpinski curve. If the finite critical points lie on the boundary of the basin of attraction of ∞, then the Julia set is a Sierpinski gasket. Investigate the special topological properties of the Julia sets for this family, learning about Sierpinski curves, gaskets and carpets in the process. Write or use a computer program to draw the Fatou and Julia set for different cases.

Sample Resources: "Sierpinski carpets and gaskets as Julia sets of rational maps," P. Blanchard, R. L. Devaney, D. Look, M. Moreno-Rocha, P. Seal, S. Siegmund and D. Uminsky, *Dynamics on the Riemann Sphere*, Eur. Math. Soc., Zürich (2006), 97–119. "Cantor and Sierpinski, Julia and Fatou: complex topology meets complex dynamics," R. L. Devaney, *Notices Amer. Math. Soc.*, Vol. 51, No. 1 (2004), 9–15.

7. **Lattès Maps**

In 1918, Lattès constructed a family of maps whose Julia sets were the entire Riemann sphere. In this case, the set of repelling periodic points is dense in the complex plane and the dynamics on the whole plane is chaotic. The construction involves discrete lattices and endomorphisms of the complex torus. A special conjugacy is used to build rational maps with the desired properties. One such conjugacy is the famous Weierstrass $\mathcal{P}$-function. Investigate, explain and derive some of the Lattès maps, learning a variety of mathematics (number theory, complex analysis, topology, algebra) in the process.

Sample Resources: Section 3.2 of Blanchard's article on complex dynamics. "On Lattès maps," J. Milnor, *Dynamics on the Riemann Sphere*, Eur. Math. Soc., Zürich (2006), 9–43.