Math 374, Dynamical Systems, Fall 2017

The Quadratic Map Q_c is Topologically Conjugate to the Shift Map σ

1 The Set Up

Recall that $Q_c(x) = x^2 + c$ is the quadratic map and that $p_+ = \frac{1}{2}(1 + \sqrt{1-4c})$ is the larger of the two fixed points. If $c < -2$, a symmetrical piece of the bottom of the graph of Q_c lies outside the square with vertices (p_+, p_+), $(-p_+, p_+)$, $(-p_+, -p_+)$ and $(p_+, -p_+)$. This follows because $Q_c(0) = c < -p_+$ for $c < -2$.

The point $-p_+$ maps to p_+ on the first iterate and is thus eventually fixed. There are two pre-images of $-p_+$, denoted α and $-\alpha$, which are eventually fixed at p_+ after two iterates. We compute that $\alpha = \sqrt{-c - p_+}$, which is real because $c < -p_+$. The open interval $A_1 = (-\alpha, \alpha)$ maps below $-p_+$ on the first iterate, then above p_+ on the next iterate, and then off to infinity as n gets larger. Consequently, we think of A_1 as the **trapdoor;** any point whose orbit eventually lands in A_1 will escape to ∞.

Let us define the following important closed intervals:

$$\begin{aligned} I &= [-p_+, p_+] \\ I_0 &= [-p_+, -\alpha] \\ I_1 &= [\alpha, p_+] \end{aligned}$$

Note that $I = I_0 \cup A_1 \cup I_1$. The open interval A_1 and all of its pre-images A_n contain all the points that escape to ∞. The sum of the length of these intervals equals the length of I. We are interested in the set of points Λ that remain in I under iteration of Q_c. As discussed in class,

$$\Lambda = \{x \in I : Q_c^n(x) \in I \ \forall n\}$$

is a Cantor set — a nonempty, closed, and totally disconnected set.

2 The Itinerary Map

Definition 2.1 (The Itinerary Map) *Suppose $x \in I$. The itinerary of x is the infinite sequence*

$$S(x) = (s_0 s_1 s_2 s_3 \ldots) \quad \textit{where} \begin{cases} s_j = 0 & \textit{if } Q_c^j(x) \in I_0, \textit{ and} \\ s_j = 1 & \textit{if } Q_c^j(x) \in I_1. \end{cases}$$

Here, we define $Q_c^0(x) = x$, so that s_0 reveals which interval x starts in. Since $x \in \Lambda$, we know that every iterate will stay in I and can never land in A_1. Thus, $Q_c^j(x)$ is always in either I_0 or I_1 for any j. This means that the sequence defined by the itinerary map will be an infinite sequence of 0's and 1's. In other words, S is function from Λ to Σ_2, the space of sequences of 0's and 1's. The reason that S is called the *itinerary map* is that each entry in the sequence $S(x)$ will tell us whether the corresponding iterate of x is to the left of the trapdoor (0) or to the right (1).

Example 2.2 *The following itineraries can be calculated easily with a good web diagram:*

$$\begin{aligned} S(p_+) &= (11111\cdots) \\ S(-p_+) &= (01111\cdots) \\ S(\alpha) &= (10111\cdots) \\ S(-\alpha) &= (00111\cdots) \\ S(p_-) &= (00000\cdots). \end{aligned}$$

Key Observation: Note that the dynamical behavior for each x-value shown (under Q_c) is identical to the dynamical behavior of the corresponding sequence $S(x)$ under the shift map. For example, p_+ is fixed under Q_c, while its itinerary $S(p_+) = (111\cdots)$ is fixed under the shift map. The point α is eventually fixed at p_+ after two iterates, while its itinerary $S(\alpha) = (10111\cdots)$ is eventually fixed at $(111\cdots)$ after two iterates of the shift map. This will *always* be the case as the map Q_c on Λ is actually *topologically conjugate* to the shift map σ on Σ_2. In other words, the dynamics of Q_c on the Cantor set Λ are equivalent to the dynamics of the shift map σ on Σ_2! This is a truly remarkable fact demonstrating the usefulness of symbolic dynamics. We can understand the complicated dynamics of Q_c by using a simple shift map on the space of sequences of 0's and 1's.

Theorem 2.3 *If $c < -2$, then Q_c on Λ is topologically conjugate to the shift map σ on Σ_2. The itinerary map $S : \Lambda \mapsto \Sigma_2$ is the conjugacy.*

3 Proof of Theorem 2.3

There are two items we must show:

1. $S \circ Q_c = \sigma \circ S$, and
2. S is a homeomorphism.

Proof of 1. Let $x \in \Lambda$ and suppose that x has itinerary $S(x) = (s_0 s_1 s_2 s_3 \cdots)$. By definition of S,

$$x \in I_{s_0}, \quad Q_c(x) \in I_{s_1}, \quad Q_c^2(x) \in I_{s_2}, \quad Q_c^3(x) \in I_{s_3}, \quad \text{etc.},$$

where $s_i \in \{0, 1\}$. Now consider the itinerary of $Q_c(x)$. This is the itinerary of the first iterate of x. Since $Q_c(x)$ starts in I_{s_1}, the first sequence in the itinerary $S(Q_c(x))$ is s_1. Then, since $Q_c^2(x) \in I_{s_2}$, the next iterate of $Q_c(x)$ lies in the interval I_{s_2}, and thus the next sequence in the itinerary of $Q_c(x)$ is s_2. Continuing in this fashion, we have

$$S(Q_c(x)) = (s_1 s_2 s_3 \cdots) = \sigma(S(x)),$$

which proves item 1. In essence, the itinerary map S is constructed to follow the orbit of points under Q_c. So the itinerary of $Q_c(x)$ is found by simply ignoring the first element in the itinerary of x, which is precisely what the shift map σ does.

Proof of 2. This is the hard part. We must show that the itinerary map S is one-to-one, onto, continuous and has a continuous inverse.

S **is one-to-one:** Suppose that $S(x) = S(y)$ for some $x, y \in \Lambda$. By contradiction, suppose that $x \neq y$. Without loss of generality, we may assume that $x < y$ and focus our attention on the interval $[x, y]$.

Since $S(x) = S(y)$, x and y have the same itineraries, so $Q_c^n(x)$ and $Q_c^n(y)$ lie in the same subinterval I_0 or I_1 for all n. Note that Q_c is a one-to-one function on either I_0 or I_1 (since we only have less than half the parabola on either of these intervals). Using the fact that the composition of one-to-one functions is still one-to-one, we know that Q_c^n maps $[x, y]$ one-to-one onto $[Q_c^n(x), Q_c^n(y)]$. This means that for each n, $[Q_c^n(x), Q_c^n(y)] \subset I_0$ or $[Q_c^n(x), Q_c^n(y)] \subset I_1$ (everything between the endpoints x and y must map injectively between the endpoints $Q_c^n(x)$ and $Q_c^n(y)$). But this means that the entire interval $[x, y] \subset \Lambda$, which contradicts the fact that Λ is totally disconnected.

S **is onto:** For this part we need to use the Nested Interval Theorem:

Theorem 3.1 (Nested Interval Theorem) *Suppose $I_n = [a_n, b_n]$ is a sequence of closed intervals with*

$$I_1 \supset I_2 \supset I_3 \supset \cdots \supset I_n \supset I_{n+1} \supset \cdots$$

and that $\lim_{n\to\infty} b_n - a_n = 0$. Then, there exists a unique point $p \in I_n \; \forall n$. In other words,

$$\bigcap_{n=1}^{\infty} I_n = \{p\}.$$

We also need to use the following notation for preimages of Q_c. Let $J \subset I$. Then

$$\begin{aligned} Q_c^{-1}(J) &= \{x \in I : Q_c(x) \in J\} \\ &= \text{all points that are mapped into } J \text{ by } Q_c, \end{aligned}$$

$$\begin{aligned} Q_c^{-n}(J) &= \{x \in I : Q_c^n(x) \in J\} \\ &= \text{all points that are mapped into } J \text{ by } Q_c^n. \end{aligned}$$

Key Fact: If J is a closed interval, then $Q_c^{-1}(J)$ is two closed (and smaller) subintervals, one of which is in I_0 and the other of which is in I_1 (see Figure 9.6) below.

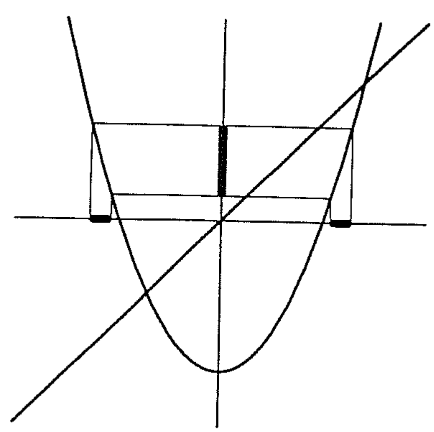

Fig. 9.6 The preimage of a closed interval J is a pair of closed intervals, one in I_0 and one in I_1.

Suppose that $s = (s_0 s_1 s_2 \cdots)$ is an arbitrary sequence in Σ_2. To show that S is onto, we must show that there exists an $x \in \Lambda$ such that $S(x) = s$. We will do this by constructing the point x as the infinite intersection of closed sets.

Define

$$\begin{aligned}
I_{s_0} &= \{x \in I : x \in I_{s_0}\} \\
I_{s_0 s_1} &= \{x \in I : x \in I_{s_0} \text{ and } Q_c(x) \in I_{s_1}\} \\
I_{s_0 s_1 s_2} &= \{x \in I : x \in I_{s_0},\ Q_c(x) \in I_{s_1}, \text{ and } Q_c^2(x) \in I_{s_2}\} \\
&\vdots \\
I_{s_0 s_1 s_2 \cdots s_n} &= \{x \in I : x \in I_{s_0},\ Q_c(x) \in I_{s_1},\ \ldots\ Q_c^n(x) \in I_{s_n}\}
\end{aligned}$$

The set $I_{s_0 s_1 s_2 \cdots s_n}$ consists of all the points in I whose first $n+1$ entries in their itinerary agree with the first $n+1$ entries of s. For example, if $s = (0110\cdots)$, then $I_{s_0 s_1 s_2 s_3} = I_{0110}$ consists of all the points that start in I_0, with their first and second iterates in I_1, and third iterate in I_0.

This set can be found by repeatedly finding pre-images under Q_c and taking their intersection. Specifically, we have that

$$I_{s_0 s_1 s_2 \cdots s_n} = I_{s_0} \cap Q_c^{-1}(I_{s_1}) \cap Q_c^{-2}(I_{s_2}) \cap \cdots \cap Q_c^{-n}(I_{s_n}),$$

by definition of Q_c^{-j}. This shows that $I_{s_0 s_1 s_2 \cdots s_n}$ is a closed set since it is the finite intersection of closed intervals. Moreover, because of the key fact above, we have

$$I_{s_0} \supset I_{s_0 s_1} \supset I_{s_0 s_1 s_2} \supset \cdots \supset I_{s_0 s_1 s_2 \cdots s_{n-1}} \supset I_{s_0 s_1 s_2 \cdots s_n},$$

a nested intersection. The length of $I_{s_0 s_1 s_2 \cdots s_n}$ is approaching 0 as $n \to \infty$ because $Q_c^{-n}(I_{s_n})$ is a smaller and smaller interval as $n \to \infty$ (Q_c is expanding so Q_c^{-1} is contracting). Applying the Nested Interval Theorem, we let

$$x = \bigcap_{n=0}^{\infty} I_{s_0 s_1 s_2 \cdots s_n}.$$

Then $x \in \Lambda$ because the nth iterate of x under Q_c lies in I_{s_n} for each n, so the orbit never escapes through the trapdoor. In addition, we have that $S(x) = (s_0 s_1 s_2 \cdots s_n \cdots) = s$ by construction, since $Q_c^n(x) \in I_{s_n}$ $\forall n$. This proves that S is onto.

S **is continuous:** Pick $x \in \Lambda$ and suppose that $S(x) = (s_0 s_1 s_2 \cdots s_n \cdots) \in \Sigma_2$. Let $\epsilon > 0$ be given and pick $n \in \mathbb{N}$ such that $1/2^n < \epsilon$. We must find a $\delta > 0$ such that $|x - y| < \delta$ implies that $d(S(x), S(y)) < \epsilon$, where d is the standard metric on Σ_2.

Since $S(x) = (s_0 s_1 s_2 \cdots s_n \cdots)$, $x \in I_{s_0 s_1 \cdots s_n}$, which is some small, closed set in I. Choose δ so that if $y \in \Lambda$ and $|x - y| < \delta$, then $y \in I_{s_0 s_1 \cdots s_n}$ as well. This is clearly possible if x is in the interior of $I_{s_0 s_1 \cdots s_n}$, because this is a closed interval with some finite (albeit small) length. We then choose δ so that the δ-neighborhood about x lies inside $I_{s_0 s_1 \cdots s_n}$. If x happens to be an endpoint of $I_{s_0 s_1 \cdots s_n}$ (which means it will eventually be fixed at p_+ under iteration), then points to one side of x will eventually escape to ∞, so we only focus on the intersection of a δ-neighborhood about x with $I_{s_0 s_1 \cdots s_n}$. Again, it is possible to choose δ sufficiently small to ensure that this intersection lies within $I_{s_0 s_1 \cdots s_n}$. Thus, if $y \in \Lambda$ and $y \in I_{s_0 s_1 \cdots s_n}$, then the first $n+1$ entries of $S(y)$ will agree with the first $n+1$ entries of $S(x)$. By the Proximity Theorem, this means that $d(S(x), S(y)) \leq 1/2^n < \epsilon$, as desired.

S^{-1} **is continuous:** This proof is left to you as a HW exercise. :)