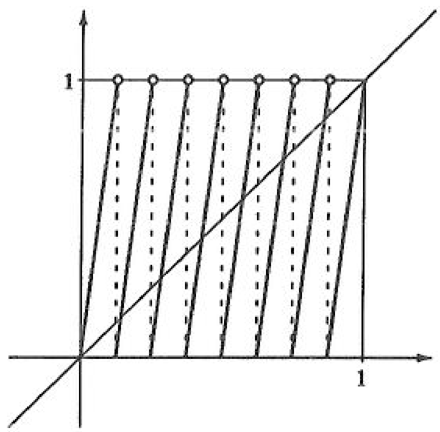
Math 374, Dynamical Systems, Spring 2014

Partial Solutions for HW #5

Chapter 10, #20
Prove that the doubling map $D : [0,1) \mapsto [0,1)$ is chaotic, where D is defined as

$$D(x) \;=\; \begin{cases} 2x & \text{if } 0 \le x < 1/2 \\ 2x-1 & \text{if } 1/2 \le x < 1. \end{cases}$$

One way to prove that D is chaotic is to consider the graph of D^n. As computed on an earlier homework, the graph of D^n consists of 2^n parallel line segments of slope 2^n, equally spaced over $[0,1)$. The domain of each line segment is a subinterval of the form $[\frac{k}{2^n}, \frac{k+1}{2^n})$, where $k \in \{0, 1, 2, \ldots, 2^n - 1\}$. The key idea we will use repeatedly is that each interval of the form $[\frac{k}{2^n}, \frac{k+1}{2^n})$ is mapped continuously onto $[0,1)$ (see figure for a graph of $D^3(x)$).



(b) $D^3(x) = 8x \bmod 1$

Periodic Points are Dense in $[0,1)$: Pick an arbitrary $y \in [0,1)$ and let $\epsilon > 0$ be given. Choose $n \in \mathbb{N}$ sufficiently large so that $1/2^n < \epsilon$. Then there exists a k such that $[\frac{k}{2^n}, \frac{k+1}{2^n}) \subset (y-\epsilon, y+\epsilon)$. Since the graph of D^n over $[\frac{k}{2^n}, \frac{k+1}{2^n})$ stretches continuously from 0 to 1, it intersects the diagonal $y = x$. Thus, there is a solution to $D^n(x) = x$ inside the interval $[\frac{k}{2^n}, \frac{k+1}{2^n})$ and therefore inside the interval $(y-\epsilon, y+\epsilon)$. This shows that there is a periodic point within ϵ of y, which proves that periodic points are dense.

Topological Transitivity: Let U and V be two arbitrary open sets in $[0,1)$. Since U is open, we can choose $n \in \mathbb{N}$ sufficiently large and choose k appropriately such that $[\frac{k}{2^n}, \frac{k+1}{2^n}) \subset U$. Since the graph of D^n over $[\frac{k}{2^n}, \frac{k+1}{2^n})$ stretches continuously from 0 to 1, it not only intersects V, it covers V completely. Thus, since a subset of U covers V, we have that $D^n(U) \cap V = V \neq \emptyset$. This proves that D is topologically transitive.

Sensitive Dependence on Initial Conditions: Let $\delta = 1/2$. Pick an arbitrary $y \in [0,1)$ and let $\epsilon > 0$ be given. We will show that there exists an $x \in [0,1)$ within ϵ of y and an $n \in \mathbb{N}$ such that the nth iterates of x and y are at least δ apart.

Once again, choose $n \in \mathbb{N}$ sufficiently large so that $1/2^n < \epsilon$. It follows that there exists a k such that $y \in [\frac{k}{2^n}, \frac{k+1}{2^n}) \subset (y - \epsilon, y + \epsilon)$. Divide the subinterval $[\frac{k}{2^n}, \frac{k+1}{2^n})$ in half. If y is in the left half of this subinterval, then $D^n(y) < 1/2$, and we choose x sufficiently close enough to $\frac{k+1}{2^n}$ so that $D^n(x)$ is sufficiently close to 1. On the other hand, if y is in the right half of this subinterval, then $D^n(y) > 1/2$, and we choose $x = \frac{k}{2^n}$ so that $D^n(x) = 0$. In either case, $|D^n(x) - D^n(y)| > \delta = 1/2$, as desired.

Note: If y happens to be at the midpoint of the subinterval $[\frac{k}{2^n}, \frac{k+1}{2^n})$, then $D^n(y) = 1/2$ and consequently $D^{n+1}(y) = 0$. In this case we just choose x slightly to the left of y, and then $D^{n+1}(x)$ is close to 1 and $|D^{n+1}(x) - D^{n+1}(y)| > \delta = 1/2$. This completes the proof that D is chaotic.

The other way to prove that D is chaotic is to show that D is conjugate to the shift map σ on Σ_2. This can be done using binary expansion. The specific conjugacy is the map that sends $x \in [0, 1)$ to the entries in its base 2 expansion. One should check that such a map is actually a homeomorphism (it is). In addition, since SDIC is not actually preserved under homeomorphism and since D is not continuous, one should also prove that D has SDIC, as shown above.

Additional Problem:

Show that the inverse of the itinerary map, S^{-1}, is continuous.

We first recall the definition of S. Suppose $x \in \Lambda$. The itinerary of x is the infinite sequence

$$S(x) \;=\; (s_0 s_1 s_2 s_3 \ldots) \quad \text{where} \begin{cases} s_j = 0 & \text{if } Q_c^j(x) \in I_0, \text{ and} \\ s_j = 1 & \text{if } Q_c^j(x) \in I_1. \end{cases}$$

The function S^{-1} maps sequences in Σ_2 back to real numbers in Λ. In other words, $S^{-1} : \Sigma_2 \mapsto \Lambda$. Given a sequence $s \in \Sigma_2$, $S^{-1}(s)$ is the point in the Cantor set Λ whose itinerary under Q_c is s.

Proof: Let $s = (s_0 s_1 s_2 \cdots s_n \cdots) \in \Sigma_2$ be an arbitrary sequence and let $\epsilon > 0$ be given. Denote $x = S^{-1}(s) \in \Lambda$ as the image of s under S^{-1}. In other words, $S(x) = s$ and the itinerary of $x \in \Lambda$ under Q_c is given by s. We must find a $\delta > 0$ such that

$$d(s, t) \;<\; \delta \quad \Longrightarrow \quad |S^{-1}(s) - S^{-1}(t)| \;<\; \epsilon.$$

Note that on the left-hand side, the metric being used is d, the distance function on Σ_2, while on the right-hand side, the metric used is simply the absolute value function as $S^{-1}(s)$ and $S^{-1}(t)$ are real numbers.

The key to the proof is the special closed interval $I_{s_0 s_1 s_2 \cdots s_n}$ used to show that S was onto. Recall that

$$I_{s_0 s_1 s_2 \cdots s_n} \;=\; \{y \in I : y \in I_{s_0},\; Q_c(y) \in I_{s_1},\; \ldots\;, Q_c^n(y) \in I_{s_n}\}.$$

The set $I_{s_0 s_1 s_2 \cdots s_n}$ consists of all the points in I whose first $n + 1$ entries in their itinerary agree with the first $n + 1$ entries of s. By definition of S^{-1}, $x \in I_{s_0 s_1 s_2 \cdots s_n}$ $\forall n$. This set can be found by repeatedly finding pre-images under Q_c and taking their intersection. Specifically, we have that

$$I_{s_0 s_1 s_2 \cdots s_n} \;=\; I_{s_0} \,\cap\, Q_c^{-1}(I_{s_1}) \,\cap\, Q_c^{-2}(I_{s_2}) \,\cap\, \cdots \,\cap\, Q_c^{-n}(I_{s_n}),$$

by definition of Q_c^{-j}. This shows that $I_{s_0 s_1 s_2 \cdots s_n}$ is a closed interval since it is the finite intersection of closed intervals. Moreover, we know that the length of $I_{s_0 s_1 s_2 \cdots s_n}$ is approaching 0 as $n \to \infty$ because $Q_c^{-n}(I_{s_n})$ is a smaller and smaller interval as $n \to \infty$ (Q_c is expanding so Q_c^{-1} is contracting).

Now, choose $n \in \mathbb{N}$ sufficiently large such that $I_{s_0 s_1 s_2 \cdots s_n} \subset (x - \epsilon, x + \epsilon)$. This is possible since the intervals $I_{s_0 s_1 s_2 \cdots s_n}$ are shrinking. Let $\delta = 1/2^n$. If $d(s,t) < \delta = 1/2^n$, then $s_i = t_i \ \forall i \leq n$ by the Proximity Theorem. But this in turn means that $S^{-1}(t) \in I_{s_0 s_1 s_2 \cdots s_n}$ since the itinerary of $S^{-1}(t)$ agrees with the itinerary of $x = S^{-1}(s)$ in the first $n+1$ entries. Since $I_{s_0 s_1 s_2 \cdots s_n} \subset (x - \epsilon, x + \epsilon)$, we have $S^{-1}(t) \in (x - \epsilon, x + \epsilon)$, which shows

$$d(s,t) < \delta \quad \Longrightarrow \quad |S^{-1}(s) - S^{-1}(t)| < \epsilon,$$

as desired.