MATH 374 Sample Final Exam Questions

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Below are some **sample** final exam questions. Collectively, these are not intended to represent an actual exam nor do they completely cover all the material that could be asked on the exam.

1. State the Repelling Fixed Point Theorem. What major theorem from mathematics is used to prove this theorem?

2. Consider the logistic map $F_\lambda(x) = \lambda x(1-x)$.

   (a) Find the fixed points in terms of the parameter λ.

   (b) For what values of λ is each fixed point attracting?

   (c) Compute $F_\lambda^2(x)$.

   (d) Show that F_λ has a period-doubling bifurcation at $\lambda = 3$.

   (e) Describe the bifurcation that occurs at $\lambda = 1$.

   (f) Sketch the bifurcation diagram for the logistic family near these key parameter values. Use solid lines for attracting cycles and dashed lines for repellers.

3. Use graphical analysis and/or calculus to describe the fate of **all** orbits for the following dynamical systems.

   (a) $F(x) = x - x^4$

   (b) $G(x) = -1/x$

   (c) $H(x) = |x-1|$

4. Prove that the map $F : \Sigma \mapsto \Sigma$ given by

$$F(s_0 s_1 s_2 s_3 \ldots) \;=\; (s_1 s_4 s_9 s_{16} \ldots)$$

   is continuous.

5. **(a)** What are the three properties of a chaotic dynamical system? Give precise mathematical definitions of each property.

   (b) Which property follows from the other two and under what circumstances?

   (c) Give three examples of chaotic dynamical systems. Prove one of them is actually chaotic.

6. Compute the Schwarzian derivative of $f(x) = x^n$. For which values of n is it true that the Schwarzian derivative is strictly negative for all $x \in \mathbb{R}$.

7. **(a)** What is Devaney's definition of a fractal?

   (b) What is the topological dimension of the Koch snowflake curve?

   (c) What is the fractal dimension of the Koch snowflake curve? Explain.

8. **(a)** Give the definition of the Julia set of a rational map.

   (b) Describe the type of dynamics occuring in the Julia set.

   (c) What is the Julia set for $Q_0(z) = z^2$?

9. Describe the fate of all orbits in the complex plane under each of the following linear maps:

   (a) $L(z) = 2iz$

   (b) $M(z) = e^{\sqrt{2}\,\pi i} z$

10. Some conceptual questions:

    (a) Suppose that the continuous function f maps the closed interval $[a, b]$ onto the closed interval $[c, d]$, with $[a, b] \subset [c, d]$. Show that f has a fixed point in $[a, b]$.

    (b) Suppose that f and g are topologically conjugate via the homeomorphism h and that f has a periodic point of prime period 6. Show that g has periodic cycles of all even periods.

    (c) Determine whether 1/7 is in the middle-thirds Cantor set.