$$< \frac{1}{2}\sum_{i=0}^{\infty}\frac{|s_i - t_i|}{2^i}$$
$$= \frac{1}{2}d[\mathbf{s},\mathbf{t}].$$

Therefore,

$$d[G(\mathbf{s}), G(\mathbf{t})] < \frac{1}{2}d[\mathbf{s},\mathbf{t}] < \frac{1}{2}\delta = \frac{1}{2^n} < \epsilon$$

upon choosing appropriate values for n and δ.

18c) $H(s_0 s_1 s_2 \ldots) = (s_1 s_0 s_3 s_2 s_5 s_4 \ldots)$.

Suppose $\mathbf{s}$ and $\mathbf{t}$ are such that $s_{2k+1} \neq t_{2k+1}$ and $s_{2k} = t_{2k}$ for all k. In other words, $\mathbf{s}$ and $\mathbf{t}$ differ with respect to every other entry starting with s_1 and t_1. Then

$$d[H(\mathbf{s}), H(\mathbf{t})] = \sum_{i=0}^{\infty}\frac{|s_{2i+1} - t_{2i+1}|}{2^{2i}}$$
$$= 2\sum_{i=0}^{\infty}\frac{|s_{2i+1} - t_{2i+1}|}{2^{2i+1}}$$
$$= 2d[\mathbf{s},\mathbf{t}].$$

For example, let $\mathbf{s} = (000\ldots)$ and $\mathbf{t} = (\overline{01})$. Then $H(\mathbf{s}) = \mathbf{s}$ and $H(\mathbf{t}) = (\overline{10})$, and so we have

$$\frac{4}{3} = d[H(\mathbf{s}), H(\mathbf{t})] = 2d[\mathbf{s},\mathbf{t}] = 2\cdot\frac{2}{3} = \frac{4}{3}.$$

Now suppose $s_{2k} \neq t_{2k}$ and $s_{2k+1} = t_{2k+1}$ for all k. These symbol sequences still differ with respect to every other entry, but offset by one position. In this case,

$$d[H(\mathbf{s}), H(\mathbf{t})] = \sum_{i=0}^{\infty}\frac{|s_{2i} - t_{2i}|}{2^{2i+1}}$$
$$= \frac{1}{2}\sum_{i=0}^{\infty}\frac{|s_{2i} - t_{2i}|}{2^{2i}}$$
$$= \frac{1}{2}d[\mathbf{s},\mathbf{t}].$$

Now, the claim is that for $\mathbf{s}, \mathbf{t} \in \Sigma$,

$$\frac{1}{2}d[\mathbf{s},\mathbf{t}] \leq d[H(\mathbf{s}), H(\mathbf{t})] \leq 2d[\mathbf{s},\mathbf{t}]. \tag{9.3}$$

If you believe this, then the continuity of H follows almost immediately. As usual, let $\epsilon > 0$ be given, and choose n so that $1/2^n < \epsilon$. Then let $\delta = 1/2^{n+1}$ and suppose $d[\mathbf{s},\mathbf{t}] < \delta$. But $d[H(\mathbf{s}), H(\mathbf{t})] \leq 2d[\mathbf{s},\mathbf{t}]$ by (9.3), and moreover,

$$2d[\mathbf{s},\mathbf{t}] < 2\delta = 1/2^n < \epsilon.$$

Thus $d[H(\mathbf{s}), H(\mathbf{t})] < \epsilon$ and so H is continuous.

18d) $J(s_0 s_1 s_2 \ldots) = (\hat{s}_0 \hat{s}_1 \hat{s}_2 \ldots)$ where $\hat{s}_j = 1$ if $s_j = 0$, and $\hat{s}_j = 0$ if $s_j = 1$.

We claim that $d[J(\mathbf{s}), J(\mathbf{t})] = d[\mathbf{s},\mathbf{t}]$. This is because $|s_j - t_j| = |\hat{s}_j - \hat{t}_j|$ which is easily seen by enumerating all possible cases (of which there are only four). The proof that J is continuous is immediate. Just choose $\delta = \epsilon$.

18e) $K(s_0 s_1 s_2 \ldots) = ((1 - s_0)(1 - s_1)(1 - s_2)\ldots)$.

We remark that the mapping K makes sense only if we think of the s_i as binary digits (which they aren't!). In this context, $K = J$ of Exercise 18d and so K is continuous since J is.

18f) $L(s_0 s_1 s_2 \ldots) = (s_0 s_2 s_4 s_6 \ldots)$.

The trick here is to pick δ small enough so that $\mathbf{s}$ and $\mathbf{t}$ agree on the first $2n + 1$ entries.

As always, let $\epsilon > 0$ be given and choose n so that $1/2^n < \epsilon$. Now let $\delta = 1/2^{2n}$ and suppose $d[\mathbf{s},\mathbf{t}] < \delta$. Then by the Proximity Theorem, $s_i = t_i$ for $i = 0, 1, \ldots, 2n$, and so,

$$d[L(\mathbf{s}), L(\mathbf{t})] = \sum_{i=n+1}^{\infty}\frac{|s_{2i} - t_{2i}|}{2^i}$$
$$\leq \sum_{i=n+1}^{\infty}\frac{1}{2^i}$$
$$= \frac{1/2^{n+1}}{1 - 1/2}$$
$$= \frac{1}{2^n} < \epsilon.$$