Similar arguments show that if $x \in (7/9, 8/9)$, then $T^n(x) \to -\infty$ as $n \to \infty$.

**14.** Let $\Gamma = \{x \in [0,1] \mid T^n(x) \in [0,1] \text{ for all } n\}$. Prove $\Gamma = K$, the Cantor middle-thirds set.

Let $x = 0.a_1a_2a_3\ldots$ and partition the unit interval so that $[0,1] = [0,1/3] \cup (1/3,2/3) \cup [2/3,1]$. One of the key ideas of the proof is that the value of $a_1$ alone determines which of these three intervals contains $x$. If $a_1 = 0$, then $x \in [0,1/3]$; if $a_1 = 1$, then $x \in (1/3,2/3)$; and if $a_1 = 2$, then $x \in [2/3,1]$. At first glance, the boundary points appear to be exceptions to this general rule since $1/3 = 0.1\overline{0}$ and $2/3 = 0.1\overline{2}$, but even these may be written as $0.0\overline{2}$ and $0.2\overline{0}$, respectively.

Another key idea is the important result of Exercise 12: all $x$ in the interval $(1/3,2/3)$ have orbits which shoot off to $-\infty$. But these are precisely those $x$ whose first ternary digit is equal to 1. Indeed, the claim is that any $x$ having *some* ternary digit equal to 1 has an orbit which escapes to $-\infty$! This is true because

$$T(x) = \begin{cases} 0.a_2a_3a_4\ldots & \text{if } 0 \le x \le 1/2 \\ 0.\hat{a}_2\hat{a}_3\hat{a}_4\ldots & \text{if } 1/2 \le x \le 1 \end{cases}$$

where

$$\hat{a}_i = \begin{cases} 0 & \text{if } a_i = 2 \\ 1 & \text{if } a_i = 1 \\ 2 & \text{if } a_i = 0 \end{cases} .$$

Note that this definition of $T$ is unambiguous at $x = 1/2$ which itself has ternary expansion $0.\overline{1}$.

Now the proof that $\Gamma = K$ goes as follows. Suppose $x \in \Gamma$. Then the ternary expansion of $x$ can not have a 1 in it, for if it did, $T^n(x) \in (1/3,2/3)$ for some $n$ and the orbit would escape to $-\infty$. Thus, $x$ is also in $K$. Conversely, suppose $x \in K$. Then the ternary expansion of $x$ again has no 1s. Thus, $T^n(x) \notin (1/3,2/3)$ for all $n$ and so the orbit of $x$ can not escape. Hence, $x$ is in $\Gamma$.

**15.** Suppose $x \in \Gamma$ has ternary expansion $0.a_1a_2a_3\ldots$. What is the ternary expansion of $T(x)$? Be careful: there are two very different cases!

By the combined results of Exercises 9 & 12, we know that either $x \in [0,1/3]$ or $x \in [2/3,1]$. If $x \in [0,1/3]$, then its leading ternary digit is 0; otherwise, it's 2. These are the two cases that must be dealt with.

If $a_1 = 0$, then $T(x) = 3x$. As we've already seen in Exercise 10, multiplying a ternary expansion by 3 simply shifts the ternary point one place to the right. If $a_1 = 2$, then $T(x) = 3 - 3x = 3(1-x)$. Finally, if $x = 0.a_1a_2a_3\ldots$, then $1 - x = 0.\hat{a}_1\hat{a}_2\hat{a}_3\ldots$ where

$$\hat{a}_i = \begin{cases} 0 & \text{if } a_i = 2 \\ 2 & \text{if } a_i = 0 \end{cases} .$$

Putting all these facts together, we have that

$$T(x) = \begin{cases} 0.a_2a_3a_4\ldots & \text{if } a_1 = 0 \\ 0.\hat{a}_2\hat{a}_3\hat{a}_4\ldots & \text{if } a_1 = 2 \end{cases}$$

for $x \in \Gamma$.