Chapter 7

The Quadratic Family

Exercises

1. List the intervals which are removed in the third and fourth stages of the construction of the Cantor middle-thirds set.

Step 1: $(1/3, 2/3)$

Step 2: $(1/9, 2/9) \cup (7/9, 8/9)$

Step 3: $(1/27, 2/27) \cup (7/27, 8/27) \cup (19/27, 20/27) \cup (25/27, 26, 27)$

Step 4:

$(1/81, 2/81) \cup (7/81, 8/81) \cup (19/81, 20/81) \cup (25/81, 26/81) \cup (55/81, 56/81) \cup (61/81, 62/81) \cup (73/81, 74/81) \cup (79/81, 80/81)$

2. Compute the sum of the lengths of all the intervals which are removed from the interval $[0, 1]$ in the construction of the Cantor middle-thirds set.

From Exercise 1, it should be clear that 2^{n-1} open intervals are removed at the nth stage of the construction of the Cantor middle-thirds set, each having width $1/3^n$. The combined length of these intervals is

$$1 \cdot \frac{1}{3} + 2 \cdot \frac{1}{9} + 4 \cdot \frac{1}{27} + \cdots + 2^{n-1} \cdot \frac{1}{3^n} + \cdots,$$

but

$$\begin{aligned} \sum_{i=1}^{\infty} \frac{2^{i-1}}{3^i} &= \sum_{i=1}^{\infty} \frac{2^i}{2 \cdot 3^i} \\ &= \frac{1}{2} \sum_{i=1}^{\infty} \left(\frac{2}{3}\right)^i \\ &= \frac{1}{2} \left(\frac{2/3}{1 - 2/3}\right) \\ &= 1. \end{aligned}$$

In other words, construction of the Cantor middle-thirds set is a subtractive process which removes an infinite number of open intervals with combined length equal to that of the original unit interval. Apparently, little is left behind at the conclusion of this process... or so it seems.

In the next five exercises,[1] find the rational numbers whose ternary expansion is given by:

3. $$\begin{aligned} 0.\overline{21} &= \left(\tfrac{2}{3} + \tfrac{1}{3^2}\right) + \left(\tfrac{2}{3^3} + \tfrac{1}{3^4}\right) + \cdots \\ &= \tfrac{7}{3^2} + \tfrac{7}{3^4} + \cdots \\ &= 7\left(\tfrac{1}{9} + \tfrac{1}{9^2} + \cdots\right) \\ &= 7\left(\tfrac{1/9}{1-1/9}\right) \\ &= 7\left(\tfrac{1/9}{8/9}\right) \\ &= \tfrac{7}{8} \end{aligned}$$

4. $$\begin{aligned} 0.\overline{022} &= \left(\tfrac{0}{3} + \tfrac{2}{3^2} + \tfrac{2}{3^3}\right) + \left(\tfrac{0}{3^4} + \tfrac{2}{3^5} + \tfrac{2}{3^6}\right) + \cdots \\ &= \tfrac{8}{3^3} + \tfrac{8}{3^6} + \cdots \\ &= 8\left(\tfrac{1}{27} + \tfrac{1}{27^2} + \cdots\right) \\ &= 8\left(\tfrac{1/27}{1-1/27}\right) \\ &= 8\left(\tfrac{1/27}{26/27}\right) \\ &= \tfrac{4}{13} \end{aligned}$$

5. $$\begin{aligned} 0.00\overline{2} &= \tfrac{2}{3^3} + \tfrac{2}{3^4} + \cdots \\ &= \tfrac{2}{3^3}\left(1 + \tfrac{1}{3} + \cdots\right) \\ &= \tfrac{2}{3^3}\left(\tfrac{1}{1-1/3}\right) \\ &= \tfrac{2}{3^3}\left(\tfrac{3}{2}\right) \\ &= \tfrac{1}{9} \end{aligned}$$

[1] In almost all cases, the technique is to group successive pairs or triplets of terms and simplify. Grouping of terms is permissible since a geometric series is absolutely convergent.