$$\begin{aligned} &\neq -25/2 \\ &= S(F \cdot G)(x) \end{aligned}$$

since $F \cdot G(x) = e^{5x}$.

4. Is it true that $S(cF)(x) = cSF(x)$ where c is a constant? If so, prove it. If not, give a counterexample.

Neither multiplicative nor additive constants have any effect on the Schwarzian derivative. Since $(cF)^{(n)} = c \cdot F^{(n)}$ for all n, we have that

$$\begin{aligned} S(cF)(x) &= \frac{cF'''(x)}{cF'(x)} - \frac{3}{2}\left[\frac{cF''(x)}{cF'(x)}\right]^2 \\ &= \frac{F'''(x)}{F'(x)} - \frac{3}{2}\left[\frac{F''(x)}{F'(x)}\right]^2 \\ &= SF(x). \end{aligned}$$

Also, since $(F+c)^{(n)} = F^{(n)}$ for all n,

$$S(F+c)(x) = SF(x).$$

5. Give an example of a function that has $SF(x) > 0$ for at least some x-values.

Exercise 4i at the end of Chapter 5 provides such an example. There we found the origin to be a weakly repelling fixed point for $F(x) = -x - x^3$ since

$$-2F'''(0) - 3[F''(0)]^2 = 12.$$

(This is the same as computing $SF(0) = 6$, by the way.) In fact,

$$\begin{aligned} SF(x) &= \frac{F'''(x)}{F'(x)} - \frac{3}{2}\left[\frac{F''(x)}{F'(x)}\right]^2 \\ &= \frac{-6}{\cdots} - \frac{3}{2}\left[\frac{-6x}{\cdots}\right]^2 \end{aligned}$$

or when

$$|x| < \frac{\sqrt{6}}{6}.$$

This example illustrates but one of the two cases mentioned in the proof of the Schwarzian Min-Max Principle given in the text.

6. Prove that $S(1/x) = 0$ and $S(ax+b) = 0$. Conclude that $SF(x) = 0$ where

$$F(x) = \frac{1}{ax+b}.$$

Let $R(x) = 1/x$ and $L(x) = ax + b$. Then $R'(x) = -1/x^2$, $R''(x) = 2/x^3$, and $R'''(x) = -6/x^4$. Also, $L'(x) = a$ and $L^{(n)}(x) = 0$ for all $n > 1$. Therefore,

$$\begin{aligned} SR(x) &= \frac{-6x^{-4}}{-x^{-2}} - \frac{3}{2}\left[\frac{2x^{-3}}{-x^{-2}}\right]^2 \\ &= \frac{6}{x^2} - \frac{6}{x^2} \\ &= 0 \end{aligned}$$

and

$$SL(x) = \frac{0}{a} - \frac{3}{2}\left[\frac{0}{a}\right]^2 = 0.$$

Applying the chain rule for Schwarzian derivatives,

$$\begin{aligned} S(R \circ L)(x) &= SR(L(x)) \cdot [L'(x)]^2 + SL(x) \\ &= 0 \cdot a^2 + 0 \\ &= 0. \end{aligned}$$

Question: Do all totally periodic functions have zero Schwarzian derivative?