$$= 6\cot(x^2+1) - 4x^2 - \frac{3}{2}\left(2x\cot(x^2+1) + \frac{1}{x}\right)^2$$
$$= 6\cot(x^2+1) - 4x^2 - \frac{3}{2}\left(4x^2\cot^2(x^2+1) + 4\cot(x^2+1) + \frac{1}{x^2}\right)$$
$$= -\left(4x^2 + 6x^2\cot^2(x^2+1) + \frac{3}{2x^2}\right)$$
$$< 0 \qquad \text{for all } x.$$

We better check our work. Let $C(x) = \cos x$ and $G(x) = x^2+1$ so that $C \circ G(x) = \cos(x^2+1)$. An easy computation yields

$$SC(x) = \frac{\sin x}{-\sin x} - \frac{3}{2}\left[\frac{-\cos x}{-\sin x}\right]^2$$
$$= -1 - \frac{3}{2}\cot^2 x$$

and

$$SG(x) = \frac{0}{2x} - \frac{3}{2}\left[\frac{2}{2x}\right]^2$$
$$= -\frac{3}{2x^2}.$$

So, by the chain rule for Schwarzian derivatives,

$$S(C\circ G)(x) = SC(G(x))\cdot[G'(x)]^2 + SG(x)$$
$$= \left(-1 - \frac{3}{2}\cot^2(x^2+1)\right)\cdot 4x^2 - \frac{3}{2x^2}$$
$$= -4x^2 - 6x^2\cot^2(x^2+1) - \frac{3}{2x^2},$$

which checks.

1e) $F(x) = \arctan x$; $F'(x) = (1+x^2)^{-1}$; and $F''(x) = -2x(1+x^2)^{-2}$.

$$F'''(x) = 8x^2(1+x^2)^{-3} - 2(1+x^2)^{-2}$$
$$2(1+x^2)^{-3}(4x^2 - (1+x^2))$$
$$2(1+x^2)^{-3}(3x^2-1).$$

$$SF(x) = \frac{F'''(x)}{F'(x)} - \frac{3}{2}\left[\frac{F''(x)}{F'(x)}\right]^2$$
$$= \frac{2(1+x^2)^{-3}(3x^2-1)}{(1+x^2)^{-1}} - \frac{3}{2}\left[\frac{-2x(1+x^2)^{-2}}{(1+x^2)^{-1}}\right]^2$$
$$= \frac{2(3x^2-1)}{(1+x^2)^2} - \frac{3}{2}\left[\frac{-2x}{1+x^2}\right]^2$$
$$= \frac{12x^2 - 4 - 12x^2}{2(1+x^2)^2}$$
$$= -\frac{2}{(1+x^2)^2}$$
$$< 0 \qquad \text{for all } x.$$

2. Is it true that $S(F+G)(x) = SF(x) + SG(x)$? If so, prove it. If not, give a counterexample.

False. Unlike ordinary differentiation, the Schwarzian derivative does not distribute over addition. Let $F(x) = G(x) = e^x$. Then

$$SF(x) + SG(x) = 2\left(\frac{e^x}{e^x} - \frac{3}{2}\left[\frac{e^x}{e^x}\right]^2\right)$$
$$= 2\left(1 - \frac{3}{2}\right)$$
$$= -1,$$

but

$$S(F+G)(x) = \frac{e^x+e^x}{e^x+e^x} - \frac{3}{2}\left[\frac{e^x+e^x}{e^x+e^x}\right]^2$$
$$= 1 - \frac{3}{2}$$
$$= -\frac{1}{2}.$$

3. Is it true that $S(F\cdot G)(x) = SF(x)\cdot G(x) + F(x)\cdot SG(x)$? If so, prove it. If not, give a counterexample.

Unfortunately, there is no product-like rule for Schwarzian derivatives. Let $F(x) = e^{2x}$ and $G(x) = e^{3x}$. By Exercise 1c, $SF(x) = -2$ and $SG(x) = -9/2$. But

$$SF(x)\cdot G(x) + F(x)\cdot SG(x) = -2e^{3x} - (9/2)e^{2x}$$